

# THE NEUMANN PROBLEM FOR THE $k$ -CAUCHY-FUETER COMPLEXES OVER $k$ -PSEUDOCONVEX DOMAINS IN $\mathbb{R}^4$ AND THE $L^2$ ESTIMATE

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**ABSTRACT.** The  $k$ -Cauchy-Fueter operators and complexes are quaternionic counterparts of the Cauchy-Riemann operator and the Dolbeault complex in the theory of several complex variables. To develop the function theory of several quaternionic variables, we need to solve the non-homogeneous  $k$ -Cauchy-Fueter equation over a domain under the compatibility condition, which naturally leads to a Neumann problem. The method of solving the  $\bar{\partial}$ -Neumann problem in the theory of several complex variables is applied to this Neumann problem. We introduce notions of  $k$ -plurisubharmonic functions and  $k$ -pseudoconvex domains, establish the  $L^2$  estimate and solve this Neumann problem over  $k$ -pseudoconvex domains in  $\mathbb{R}^4$ . Namely, we get a vanishing theorem for the first cohomology groups of the  $k$ -Cauchy-Fueter complex over such domains.

## 1. INTRODUCTION

The  $k$ -Cauchy-Fueter operators,  $k = 0, 1, \dots$ , are quaternionic counterparts of the Cauchy-Riemann operator in complex analysis. The  $k$ -Cauchy-Fueter complexes on multidimensional quaternionic space, which play the role of Dolbeault complex in the theory of several complex variables, are now explicitly known [21] (cf. also [2] [4] and in particular [3] [8] [9] for  $k = 1$ ). It is quite interesting to develop the function theory of several quaternionic variables by analyzing these complexes, as it has been done for the Dolbeault complex. A well known theorem in the theory of several complex variables states that the first Dolbeault cohomology of a domain vanishes if and only if it is pseudoconvex. Many remarkable results about holomorphic functions can be deduced by considering the non-homogeneous  $\bar{\partial}$ -equation, which leads to the study of  $\bar{\partial}$ -Neumann problem (cf., e.g., [5] [7] [11]). Even on one dimensional quaternionic space, i.e.  $\mathbb{R}^4$ , the  $k$ -Cauchy-Fueter complexes

$$(1.1) \quad 0 \rightarrow C^\infty(\Omega, \odot^k \mathbb{C}^2) \xrightarrow{\mathcal{D}_0^{(k)}} C^\infty(\Omega, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2) \xrightarrow{\mathcal{D}_1^{(k)}} C^\infty(\Omega, \odot^{k-2} \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^2) \rightarrow 0,$$

$k = 2, 3, \dots$ , are nontrivial, where  $\Omega$  is a domain in  $\mathbb{R}^4$ ,  $\odot^p \mathbb{C}^2$  is the  $p$ -th symmetric power of  $\mathbb{C}^2$  and  $\Lambda^2 \mathbb{C}^2$  is 2-th exterior power of  $\mathbb{C}^2$ . The first operator  $\mathcal{D}_0^{(k)}$  is called the  $k$ -Cauchy-Fueter operator. The non-homogeneous  $k$ -Cauchy-Fueter equation

$$(1.2) \quad \mathcal{D}_0^{(k)} u = f,$$

over a domain  $\Omega$  is overdetermined, and only can be solved under the *compatibility condition*

$$(1.3) \quad \mathcal{D}_1^{(k)} f = 0.$$

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It is known [6] that (1.2)-(1.3) is solvable over a bounded domain  $\Omega$  in  $\mathbb{R}^4$  with smooth boundary when  $f$  is orthogonal to the *first cohomology group* of the  $k$ -Cauchy-Fueter complex

$$H_{(k)}^1(\Omega) := \left\{ f \in C^\infty(\overline{\Omega}; \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2); \mathcal{D}_1^{(k)} f = 0 \right\} / \left\{ \mathcal{D}_0^{(k)} u; u \in C^\infty(\overline{\Omega}; \odot^k \mathbb{C}^2) \right\},$$

which is finite dimensional, where  $\overline{\Omega}$  is the closure of  $\Omega$ . We also know the solvability of (1.2)-(1.3) over  $\mathbb{R}^{4n}$  [21] [23], from which we can drive Hartogs' phenomenon for  $k$ -regular functions, i.e. functions annihilated by the  $k$ -Cauchy-Fueter operator. The purpose of this paper is to solve (1.2)-(1.3) under certain geometric conditions for the domain  $\Omega$ , i.e. to obtain a vanishing theorem for the first cohomology group of the  $k$ -Cauchy-Fueter complex. See also [24] for a vanishing theorem for the  $k$ -Cauchy-Fueter complexes over curved compact quaternionic Kähler manifolds with negative scalar curvature.

In the quaternionic case, we have a family of operators acting on  $\odot^k \mathbb{C}^2$ -valued functions, because the group of unit quaternions,  $SU(2)$ , has a family of irreducible representations  $\odot^k \mathbb{C}^2$ ,  $k = 0, 1, \dots$ , while the group of unit complex numbers,  $S^1$ , has only one irreducible representation. The  $k$ -Cauchy-Fueter operators over  $\mathbb{R}^4$  also have the origin in physics: they are the elliptic version of *spin  $k/2$  massless field* operators (cf. e.g. [6] [10] [15] [16]) over the Minkowski space:  $\mathcal{D}_0^{(1)} \phi = 0$  corresponds to the Dirac-Weyl equation whose solutions correspond to neutrinos;  $\mathcal{D}_0^{(2)} \phi = 0$  corresponds to the Maxwell equation whose solutions correspond to photons;  $\mathcal{D}_0^{(3)} \phi = 0$  corresponds to the Rarita-Schwinger equation;  $\mathcal{D}_0^{(4)} \phi = 0$  corresponds to linearized Einstein's equation whose solutions correspond to weak gravitational fields; etc.

The main difference between the  $k$ -Cauchy-Fueter complexes and Dolbeault complex is that there exist symmetric forms except for exterior forms. Analysis of exterior forms is classical, while analysis of symmetric forms is relatively new. We can handle such forms by using two-component notation. Such notation is used by physicists as two-spinor notation for the massless field operators (cf. e.g. [15] [16] and references therein). It also appears in the study of quaternionic manifolds (cf. e.g. [24] and references therein). We will use complex vector fields in two-component notation:

$$(1.4) \quad (Z_{AA'}) := \begin{pmatrix} Z_{00'} & Z_{01'} \\ Z_{10'} & Z_{11'} \end{pmatrix} := \begin{pmatrix} \partial_{x_1} + \mathbf{i}\partial_{x_2} & -\partial_{x_3} - \mathbf{i}\partial_{x_4} \\ \partial_{x_3} - \mathbf{i}\partial_{x_4} & \partial_{x_1} - \mathbf{i}\partial_{x_2} \end{pmatrix},$$

where  $A = 0, 1$ ,  $A' = 0', 1'$ , which are motivated by the embedding of the quaternion algebra into the algebra of complex  $2 \times 2$ -matrices:

$$x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \mapsto \begin{pmatrix} x_1 + \mathbf{i}x_2 & -x_3 - \mathbf{i}x_4 \\ x_3 - \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{pmatrix}.$$

In this paper, the method of solving the  $\bar{\partial}$ -Neumann problem is extended to solve the corresponding Neumann problem for the  $k$ -Cauchy-Fueter complexes. Let us introduce notions for the model case  $k = 2$ . For simplicity, we will drop the superscript  $(k)$ . Given a nonnegative measurable function  $\varphi$ , called a *weighted function*, consider the Hilbert space  $L_\varphi^2(\Omega, \mathbb{C})$  with the weighted inner product

$$(1.5) \quad (a, b)_\varphi := \int_\Omega a \bar{b} e^{-\varphi} dV,$$

where  $dV$  is the Lebesgue measure on  $\mathbb{R}^4$ . By definition,  $\odot^2 \mathbb{C}^2$  is a subspace of  $\otimes^2 \mathbb{C}^2$ , and an element  $u$  of  $L_\varphi^2(\Omega, \odot^2 \mathbb{C}^2)$  is given by a 4-tuple  $(u_{A'B'})_\varphi$  such that  $u_{A'B'} = u_{B'A'}$ , where  $A', B' = 0', 1'$ . Its weighted  $L^2$  inner product is induced from that of  $L_\varphi^2(\Omega, \otimes^2 \mathbb{C}^2)$  by

$$\langle u, v \rangle_\varphi = \sum_{A', B' = 0', 1'} (u_{A'B'}, v_{A'B'})_\varphi = (u_{0'0'}, v_{0'0'})_\varphi + 2(u_{0'1'}, v_{0'1'})_\varphi + (u_{1'1'}, v_{1'1'})_\varphi,$$

and the weighted norm  $\|u\|_\varphi := \langle u, u \rangle_\varphi^{\frac{1}{2}}$ ; An element  $f$  of  $L_\varphi^2(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2)$  has 4 components  $f_{0'0}, f_{1'0}, f_{0'1}$  and  $f_{1'1}$ . An element  $F \in L_\varphi^2(\Omega, \Lambda^2 \mathbb{C}^2)$  has components  $F_{AB}$  with  $F_{AB} = -F_{BA}$  (only  $F_{01} = -F_{10}$  is nonzero). We need to consider  $\mathcal{D}_0$  and  $\mathcal{D}_1$  as densely defined operators between Hilbert spaces

$$(1.6) \quad L_\varphi^2(\Omega, \odot^2 \mathbb{C}^2) \xrightarrow{\mathcal{D}_0} L_\varphi^2(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2) \xrightarrow{\mathcal{D}_1} L_\varphi^2(\Omega, \Lambda^2 \mathbb{C}^2),$$

given by

$$(1.7) \quad \begin{aligned} (\mathcal{D}_0 u)_{A'A} &:= \sum_{B'=0',1'} Z_A^{B'} u_{B'A'}, & \text{for } u \in C^1(\Omega, \odot^2 \mathbb{C}^2) \\ (\mathcal{D}_1 f)_{AB} &:= \sum_{B'=0',1'} Z_{[A}^{B'} f_{B]B'}, & \text{for } f \in C^1(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2), \end{aligned}$$

where  $A, B = 0, 1, A' = 0', 1'$ . Complex vector fields  $Z_A^{A'}$ 's and  $Z_{A'}^A$ 's are obtained from  $Z_{AA'}$ 's by raising indices (cf. section 2.1). Here and in the sequel,

$$(1.8) \quad H_{[AB]} := \frac{1}{2}(H_{AB} - H_{BA}), \quad H_{(A'B')} := \frac{1}{2}(H_{A'B'} + H_{B'A'})$$

are antisymmetrisation and symmetrisation, respectively.  $\mathcal{D}_1 \circ \mathcal{D}_0 = 0$  can be checked directly (cf. (2.19)).

For a  $C^2$  real function  $\varphi$ , define

$$(1.9) \quad \mathcal{L}_2(\varphi; \xi)(x) := -2 \sum_{A,B=0,1} \sum_{A',B'=0',1'} Z_B^{A'} Z_{(A'}^A \varphi(x) \cdot \xi_{B')A} \overline{\xi_{B'B}}$$

for any  $(\xi_{A'A}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ , which plays the role of the complex Hessian in our case. A  $C^2$  function  $\varphi$  on  $\Omega$  is called *2-plurisubharmonic* if there exists a constant  $c \geq 0$  ( $c > 0$ ) such that

$$\mathcal{L}_2(\varphi; \xi)(x) \geq c|\xi|^2, \quad x \in \Omega,$$

for any  $\xi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ , where  $|\xi|^2 := \sum_{A',A} |\xi_{A'A}|^2$ . A domain  $\Omega$  is called (*strictly*) *2-pseudoconvex* if there exists a defining function  $r$  and a constant  $c \geq 0$  ( $c > 0$ ) such that

$$(1.10) \quad \mathcal{L}_2(r; \xi)(x) \geq c|\xi|^2, \quad x \in \partial\Omega,$$

for any  $(\xi_{A'A}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$  satisfying

$$(1.11) \quad \sum_{A=0,1} Z_{(A'}^A r(x) \cdot \xi_{B')A} = 0, \quad x \in \partial\Omega,$$

where  $A', B' = 0', 1'$ . It plays the role of the Levi form in several complex variables. We will show that 2-pseudoconvexity of a domain is independent of the choice of the defining function (cf. Proposition 3.4). The space of vector  $\xi \in \mathbb{C}^2 \otimes \mathbb{C}^2$  satisfying (1.11) is usually one dimensional, since there are 3 equations in (1.11) with  $(A'B') = (0'0'), (0'1') = (1'0'), (1'1')$ . Note that 2-plurisubharmonic functions and 2-pseudoconvex domains are abundant since  $\chi_1(x_1^2 + x_2^2)$ ,  $\chi_2(x_3^2 + x_4^2)$  and their sum are (strictly) 2-plurisubharmonic for any increasing smooth (strictly) convex functions  $\chi_1$  and  $\chi_2$  over  $[0, \infty)$  (cf. Proposition 3.3), and a small perturbation of a strictly 2-plurisubharmonic function is still strictly 2-plurisubharmonic.

Consider the *associated Laplacian operator*  $\square_\varphi : L_\varphi^2(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2) \longrightarrow L_\varphi^2(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2)$  given by

$$(1.12) \quad \square_\varphi := \mathcal{D}_0 \mathcal{D}_0^* + \mathcal{D}_1^* \mathcal{D}_1,$$

where  $\mathcal{D}_0^*$  and  $\mathcal{D}_1^*$  are the adjoint operators of densely defined operators  $\mathcal{D}_0$  and  $\mathcal{D}_1$ , respectively, and

$$\text{Dom}(\square_\varphi) := \{f \in L_\varphi^2(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2); f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1), \mathcal{D}_0^* f \in \text{Dom}(\mathcal{D}_0), \mathcal{D}_1 f \in \text{Dom}(\mathcal{D}_1^*)\}.$$

We can show that  $C^\infty(\overline{\Omega}, \mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$  is dense in  $\text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$  by the density Lemma 4.1, and  $f \in C^1(\overline{\Omega}, \mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$  if and only if

$$(1.13) \quad \sum_{A=0,1} Z_{(A'r \cdot f_{B'})_A}^A = 0$$

on the boundary  $\partial\Omega$ . Moreover,  $F \in C^1(\overline{\Omega}, \wedge^2 \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_1^*)$  if and only if  $F = 0$ . So we need to consider the Neumann problem

$$(1.14) \quad \begin{cases} \square_\varphi f = g, & \text{in } \Omega, \\ \sum_{A=0,1} Z_{(A'r \cdot f_{B'})_A}^A = 0, & \text{on } \partial\Omega, \\ \sum_{B'=0',1'} Z_{[A}^{B'} f_{B']} = 0, & \text{on } \partial\Omega. \end{cases}$$

See section 5 for corresponding notions for general  $k$ . The key step is to establish the following  $L^2$  estimate.

**Theorem 1.1.** *For fixed  $k \in \{2, 3, \dots\}$ , let  $\Omega$  be a bounded  $k$ -pseudoconvex domain with smooth boundary and let  $\varphi$  be a smooth strictly  $k$ -plurisubharmonic function satisfying*

$$(1.15) \quad \mathcal{L}_k(\varphi; \xi)(x) \geq c|\xi|^2, \quad x \in \Omega,$$

for some  $c > 0$  and any  $\xi \in \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2$ . Suppose that

$$(1.16) \quad C_0 := \frac{c - 4\|d\varphi\|_\infty^2}{2k + 6} > 0,$$

where  $\|d\varphi\|_\infty^2 = \sum_{j=1}^4 \|\frac{\partial \varphi}{\partial x_j}\|_{L^\infty(\Omega)}^2$ . Then the  $L^2$ -estimate

$$(1.17) \quad C_0 \|f\|_\varphi^2 \leq \|\mathcal{D}_0^* f\|_\varphi^2 + \|\mathcal{D}_1 f\|_\varphi^2$$

holds for any  $f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$ .

In particular, if  $\varphi$  only satisfies the condition (1.15), then  $\kappa\varphi$  for  $0 < \kappa < \frac{c}{4\|d\varphi\|_\infty^2}$  is a weight function satisfying the assumption (1.15)-(1.16) in the above theorem with suitable constants (cf. Remark 4.1 (1)). The  $k$ -Bergman space with respect to weight  $\varphi$  is then defined as

$$A_{(k)}^2(\Omega, \varphi) := \{f \in L_\varphi^2(\Omega, \odot^k \mathbb{C}^2); \mathcal{D}_0 f = 0\}.$$

It is infinite dimensional [14] because  $k$ -regular polynomials are in this space for bounded  $\Omega$ .

**Theorem 1.2.** *Let the domain  $\Omega$  and the weight function  $\varphi$  satisfy assumptions in the above theorem. Then*

- (1)  $\square_\varphi$  has a bounded, self-adjoint and non-negative inverse  $N_\varphi$  such that

$$\|N_\varphi f\|_\varphi \leq \frac{1}{C_0} \|f\|_\varphi, \quad \text{for any } f \in L_\varphi^2(\Omega, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2).$$

- (2)  $\mathcal{D}_0^* N_\varphi f$  is the canonical solution operator to the nonhomogeneous  $k$ -Cauchy-Fueter equation (1.2)-(1.3), i.e. if  $f$  is  $\mathcal{D}_1$ -closed, then

$$\mathcal{D}_0 \mathcal{D}_0^* N_\varphi f = f$$

and  $\mathcal{D}_0^* N_\varphi f$  orthogonal to  $A_{(k)}^2(\Omega, \varphi)$ . Moreover,

$$(1.18) \quad \|\mathcal{D}_0^* N_\varphi f\|_\varphi^2 + \|\mathcal{D}_1 N_\varphi f\|_\varphi^2 \leq \frac{1}{C_0} \|f\|_\varphi^2.$$

Since the  $k$ -Bergman space  $A_{(k)}^2(\Omega, \varphi)$  is a closed Hilbert subspace, we have the orthogonal projection  $P : L_\varphi^2(\Omega, \odot^k \mathbb{C}^2) \longrightarrow A_{(k)}^2(\Omega, \varphi)$ , the  $k$ -Bergman projection. It follows from the above theorem that

$$(1.19) \quad Pf = f - \mathcal{D}_0^* N_\varphi \mathcal{D}_0 f$$

for  $f \in \text{Dom}(\mathcal{D}_0)$ , as in the theory of several complex variables (cf. theorem 4.4.5 in [7]).

In section 2, we give the necessary preliminaries on raising or lowering primed or unprimed indices, symmetrisation and antisymmetrisation of indices, the  $k$ -Cauchy-Fueter operators, the complex vector field  $Z_A^{A'}$ 's and their formal adjoint operators, etc.. In section 3, we derive the Neumann boundary condition, introduce notions of 2-plurisubharmonic functions and 2-pseudoconvex domains, and show their properties mentioned above. In section 4.1, the  $L^2$  estimate in Theorem 1.1 is established for the model case  $k = 2$ . In section 4.2, we show the density lemma from a general result due to Hörmander, and derive Theorem 1.2 from the  $L^2$  estimate in Theorem 1.1. For general  $k$ , the Neumann boundary condition and notions of  $k$ -plurisubharmonicity and  $k$ -pseudoconvexity are introduced in section 5.1. At last, the  $L^2$  estimate in Theorem 1.1 for general  $k$  is established in section 5.2.

## 2. PRELIMINARY

**2.1. The case  $k = 2$ .** Recall that an element  $f$  of  $L_\varphi^2(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2)$  has 4 components  $f_{0'0}, f_{1'0}, f_{0'1}$  and  $f_{1'1}$  and the weighted inner product of  $L_\varphi^2(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2)$  is

$$\langle f, g \rangle_\varphi = \sum_{A=0,1} \sum_{A'=0',1'} (f_{A'A}, g_{A'A})_\varphi.$$

An element  $F \in L_\varphi^2(\Omega, \Lambda^2 \mathbb{C}^2)$  has components  $F_{AB}$  with  $F_{AB} = -F_{BA}$  (only  $F_{01} = -F_{10}$  nontrivial) and the weighted inner product of  $L_\varphi^2(\Omega, \Lambda^2 \mathbb{C}^2)$  is

$$\langle F, G \rangle_\varphi = \sum_{A,B=0,1} (F_{AB}, G_{AB})_\varphi = 2(F_{01}, G_{01})_\varphi.$$

We use

$$(2.1) \quad (\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon^{A'B'}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to raise or lower primed indices, where  $(\varepsilon^{A'B'})$  is the inverse of  $(\varepsilon_{A'B'})$ . For example,

$$f_{\dots}^{A'} = \sum_{B'=0',1'} f_{\dots B' \dots} \varepsilon^{B' A'}, \quad \sum_{A'=0',1'} f_{\dots}^{A'} \varepsilon_{A' C'} = f_{\dots C' \dots}.$$

Since  $\sum_{B'=0',1'} \varepsilon_{A' B'} \varepsilon^{B' C'} = \delta_{A'}^{C'} = \sum_{B'=0',1'} \varepsilon^{C' B'} \varepsilon_{B' A'}$ , it is the same when an index is raised (or lowered) and then lowered (or raised). Similarly we use

$$(2.2) \quad (\epsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\epsilon^{AB}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to raise or lower unprimed indices. One advantage of using raising indices is that the formal adjoint operator of  $Z_A^{A'}$  can be written in a very simple form.

**Proposition 2.1.** (1) The formal adjoint operator  $Z_\varphi^*$  of a complex vector field  $Z$  with respect to the weighted inner product (1.5) is  $Z_\varphi^* = -\overline{Z} + \overline{Z}\varphi$ .

(2) We have

$$(2.3) \quad \overline{Z_A^{A'}} = -Z_{A'}^A.$$

*Proof.* (1)  $(Za, b)_\varphi = (a, Z_\varphi^* b)_\varphi$  holds for any  $a, b \in C_0^1(\Omega, \mathbb{C})$ , because we have

$$0 = \int_\Omega Z(a\bar{b}e^{-\varphi})dV = \int_\Omega Za \cdot \bar{b}e^{-\varphi}dV + \int_\Omega a \cdot \overline{Zb} \cdot e^{-\varphi}dV - \int_\Omega ab\overline{Z\varphi} \cdot e^{-\varphi}dV.$$

(2) Note that  $Z_{A'}^{A'} = \sum_{B'=0',1'} Z_{AB'} \varepsilon^{B'A'} = Z_{A0'} \varepsilon^{0'A'} + Z_{A1'} \varepsilon^{1'A'}$  by definition. It is direct to see that

$$(2.4) \quad \begin{pmatrix} Z_{A'}^{A'} \end{pmatrix} = \begin{pmatrix} Z_{0'}^{0'} & Z_{0'}^{1'} \\ Z_{1'}^{0'} & Z_{1'}^{1'} \end{pmatrix} = \begin{pmatrix} Z_{00'} & Z_{01'} \\ Z_{10'} & Z_{11'} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} Z_{01'} & -Z_{00'} \\ Z_{11'} & -Z_{10'} \end{pmatrix}$$

and

$$(2.5) \quad \begin{pmatrix} Z_{A'}^A \end{pmatrix} = \begin{pmatrix} Z_{0'}^0 & Z_{0'}^1 \\ Z_{1'}^0 & Z_{1'}^1 \end{pmatrix} = \begin{pmatrix} Z_{10'} \epsilon^{10} & Z_{00'} \epsilon^{01} \\ Z_{11'} \epsilon^{10} & Z_{01'} \epsilon^{01} \end{pmatrix} = \begin{pmatrix} Z_{10'} & -Z_{00'} \\ Z_{11'} & -Z_{01'} \end{pmatrix}$$

by  $\epsilon^{10} = -\epsilon^{01} = 1$ ,  $\epsilon^{00} = \epsilon^{11} = 0$ . It follows from definition (1.4) of  $Z_{AA'}$ 's that

$$\overline{Z_{00'}} = Z_{11'}, \quad \overline{Z_{10'}} = -Z_{01'}, \quad \overline{Z_{11'}} = Z_{00'},$$

by which we see that (2.4)-(2.5) implies (2.3).  $\square$

For a fixed weight  $\varphi$ , we introduce the differential operator

$$\delta_{A'}^A a := Z_{A'}^A a - Z_{A'}^A \varphi \cdot a$$

for a scalar function  $a$ . Then by Proposition 2.1, the formal adjoint operator of  $Z_{A'}^{A'}$  is  $(Z_{A'}^{A'})_\varphi^* = \delta_{A'}^A$ , i.e. for any  $a, b \in C_0^1(\Omega, \mathbb{C})$  we have

$$(2.6) \quad \left( Z_{A'}^{A'} a, b \right)_\varphi = (a, \delta_{A'}^A b)_\varphi.$$

For a defining function  $r$  of the domain  $\Omega$  with  $|dr| = 1$  on the boundary and a complex vector field  $Z$ , we have Stokes' formula

$$\int_\Omega Za \cdot \bar{b}e^{-\varphi}dV + \int_\Omega a \cdot \overline{Zb} \cdot e^{-\varphi}dV - \int_\Omega a\bar{b} \cdot Z\varphi \cdot e^{-\varphi}dV = \int_{\partial\Omega} Zr \cdot a\bar{b}e^{-\varphi}dS$$

for any  $a, b \in C^1(\overline{\Omega}, \mathbb{C})$ , where  $dS$  is the surface measure of the boundary, i.e.

$$(2.7) \quad (Za, b)_\varphi = (a, Z_\varphi^* b)_\varphi + \int_{\partial\Omega} Zr \cdot a\bar{b}e^{-\varphi}dS.$$

In particular, we have

$$(2.8) \quad \left( Z_{A'}^{A'} a, b \right)_\varphi = (a, \delta_{A'}^A b)_\varphi + \mathcal{B}, \quad \mathcal{B} = \int_{\partial\Omega} Z_{A'}^{A'} r \cdot a\bar{b}e^{-\varphi}dS = - \int_{\partial\Omega} a \cdot \overline{Z_{A'}^{A'} r} \cdot b e^{-\varphi}dS,$$

by using (2.3), and by taking conjugate,

$$(2.9) \quad \left( \delta_{A'}^A a, b \right)_\varphi = \left( a, Z_{A'}^{A'} b \right)_\varphi + \mathcal{B}', \quad \mathcal{B}' = \int_{\partial\Omega} Z_{A'}^A r \cdot a\bar{b}e^{-\varphi}dS = - \int_{\partial\Omega} a \cdot \overline{Z_{A'}^A r} \cdot b e^{-\varphi}dS.$$

The following properties of symmetrisation and antisymmetrisation of indices are frequently used later.

**Lemma 2.1.** (cf. Lemma 2.1 in [23])

(1) For any  $(h_{A'B'}) \in \odot^2 \mathbb{C}^2$  and  $(H_{A'B'}) \in \otimes^2 \mathbb{C}^2$ ,

$$(2.10) \quad \sum_{A', B'=0', 1'} h_{A'B'} \overline{H_{A'B'}} = \sum_{A', B'=0', 1'} h_{A'B'} \overline{H_{(A'B')}}.$$

(2) For any  $(h_{AB}) \in \wedge^2 \mathbb{C}^2$  and  $(H_{AB}) \in \otimes^2 \mathbb{C}^2$ ,

$$(2.11) \quad \sum_{A,B=0,1} h_{AB} \overline{H_{AB}} = \sum_{A,B=0,1} h_{AB} \overline{H_{[AB]}}.$$

(3) For any  $(h_{AB}), (H_{AB}) \in \otimes^2 \mathbb{C}^2$ ,

$$(2.12) \quad \sum_{A,B=0,1} h_{BA} \overline{H_{AB}} = \sum_{A,B=0,1} h_{AB} \overline{H_{AB}} - 2 \sum_{A,B} h_{[AB]} \overline{H_{[AB]}}.$$

*Proof.* (1) This is because

$$\sum_{A',B'} h_{A'B'} \overline{H_{(A'B')}} = \frac{1}{2} \sum_{A',B'} h_{A'B'} (\overline{H_{A'B'}} + \overline{H_{B'A'}}) = \sum_{A',B'} h_{A'B'} \overline{H_{A'B'}}$$

by definition (1.8) of symmetrisation, relabeling indices and  $h_{A'B'} = h_{B'A'}$ .

(2) This is because

$$\sum_{A,B} h_{AB} \overline{H_{[AB]}} = \frac{1}{2} \sum_{A,B} h_{AB} (\overline{H_{AB}} - \overline{H_{BA}}) = \sum_{A,B} h_{AB} \overline{H_{AB}}$$

by definition (1.8) of antisymmetrisation, relabeling indices and  $h_{BA} = -h_{AB}$ .

(3) This is because

$$\sum_{A,B} h_{BA} \overline{H_{AB}} = \sum_{A,B} h_{AB} \overline{H_{AB}} + \sum_{A,B} (h_{BA} - h_{AB}) \overline{H_{AB}}$$

and the second term in the right hand side is  $-2 \sum_{A,B} h_{[AB]} \overline{H_{AB}} = -2 \sum_{A,B} h_{[AB]} \overline{H_{[AB]}}$  by using (2.11).  $\square$

### Lemma 2.2.

$$(2.13) \quad \sum_{A=0,1} f_{\dots}^A \dots A \dots = -f_{\dots 0 \dots 1 \dots} + f_{\dots 1 \dots 0 \dots}, \quad \sum_{A'=0',1'} f_{\dots}^{A'} \dots A' \dots = -f_{\dots 0' \dots 1' \dots} + f_{\dots 1' \dots 0' \dots}.$$

*Proof.* Because  $\sum_{A=0,1} f_{\dots}^A \dots A \dots = \sum_{A,B=0,1} f_{\dots B \dots A \dots} \epsilon^{BA}$  and  $\epsilon^{10} = -\epsilon^{01} = 1$ ,  $\epsilon^{00} = \epsilon^{11} = 0$ .  $\square$

This lemma means that contraction of indices is just antisymmetrisation:

$$f_A^A = -2f_{[01]}, \quad f_{A'}^{A'} = -2f_{[0'1']},$$

which is very important to establish our  $L^2$  estimate and only holds in dimension 4 (cf. Remark 4.1).

**2.2. The case for general  $k$ .** Recall that the *symmetric power*  $\odot^p \mathbb{C}^2$  is a subspace of  $\otimes^p \mathbb{C}^2$ , and an element of  $\odot^p \mathbb{C}^2$  is given by a  $2^p$ -tuple  $(f_{A'_1 \dots A'_p}) \in \otimes^p \mathbb{C}^2$  with  $A'_1 \dots A'_p = 0', 1'$  such that  $f_{A'_1 \dots A'_p}$  is invariant under permutations of subscripts, i.e.

$$f_{A'_1 \dots A'_p} = f_{A'_{\sigma(1)} \dots A'_{\sigma(p)}}$$

for any  $\sigma \in S_p$ , the group of permutations of  $p$  letters. We will use symmetrisation of indices

$$(2.14) \quad f_{\dots (A'_1 \dots A'_p) \dots} := \frac{1}{p!} \sum_{\sigma \in S_p} f_{\dots A'_{\sigma(1)} \dots A'_{\sigma(p)} \dots}$$

For  $f, g \in L^2_{\varphi}(\Omega, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2)$ , define

$$(2.15) \quad \langle f, g \rangle_{\varphi} = \sum_{A=0,1} \sum_{A'_2, \dots, A'_k=0', 1'} \left( f_{A'_2 \dots A'_k A}, g_{A'_2 \dots A'_k A} \right)_{\varphi},$$

and  $\|f\|_\varphi = \langle f, f \rangle_\varphi^{\frac{1}{2}}$ . Similarly, we define the weighted inner products of  $L_\varphi^2(\Omega, \odot^k \mathbb{C}^2)$  and  $L_\varphi^2(\Omega, \odot^{k-2} \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^2)$  as subspaces of  $L_\varphi^2(\Omega, \otimes^k \mathbb{C}^2)$ . The following proposition is the general version of Lemma 2.1 (1).

**Proposition 2.2.** *For any  $g, G \in \otimes^p \mathbb{C}^2$ , we have*

$$(2.16) \quad \sum_{B'_1, \dots, B'_p=0', 1'} g_{(B'_1 \dots B'_p)} \overline{G_{(B'_1 \dots B'_p)}} = \sum_{B'_1, \dots, B'_p=0', 1'} g_{(B'_1 \dots B'_p)} \overline{G_{B'_1 \dots B'_p}}.$$

*Proof.* By definition (2.14) of symmetrisation and relabeling indices, we have

$$L.H.S. = \frac{1}{p!} \sum_{B'_1, \dots, B'_p} \sum_{\sigma \in S_p} g_{(B'_1 \dots B'_p)} \overline{G_{B'_{\sigma(1)} \dots B'_{\sigma(p)}}} = \frac{1}{p!} \sum_{\sigma \in S_p} \sum_{B'_1, \dots, B'_p} g_{(B'_{\sigma^{-1}(1)} \dots B'_{\sigma^{-1}(p)})} \overline{G_{B'_1 \dots B'_p}},$$

which equals to R.H.S. by  $g_{(B'_{\sigma^{-1}(1)} \dots B'_{\sigma^{-1}(p)})} = g_{(B'_1 \dots B'_p)}$ .  $\square$

We need to consider  $\mathcal{D}_0$  and  $\mathcal{D}_1$  as densely defined operators between Hilbert spaces

$$(2.17) \quad L_\varphi^2(\Omega, \odot^k \mathbb{C}^2) \xrightarrow{\mathcal{D}_0} L_\varphi^2(\Omega, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2) \xrightarrow{\mathcal{D}_1} L_\varphi^2(\Omega, \odot^{k-2} \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^2),$$

given by

$$(2.18) \quad \begin{aligned} (\mathcal{D}_0 u)_{A'_2 \dots A'_k A} &:= \sum_{A'=0', 1'} Z_A^{A'} u_{A' A'_2 \dots A'_k}, & \text{for } u \in C^1(\Omega, \odot^k \mathbb{C}^2), \\ (\mathcal{D}_1 f)_{A'_3 \dots A'_k AB} &:= \sum_{A'=0', 1'} Z_{[A}^{A'} f_{B] A' A'_3 \dots A'_k}, & \text{for } f \in C^1(\Omega, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2), \end{aligned}$$

where  $A, B = 0, 1$ ,  $A'_2, \dots, A'_k = 0', 1'$ . Here and in the sequel we will use the notation

$$f_{AA'_2 \dots A'_k} := f_{A'_2 \dots A'_k A}$$

for convenience. In particular,  $f_{AA'} := f_{A' A}$ . By definition (2.18),  $(\mathcal{D}_1 \psi)_{\dots AA} = 0$ ,  $(\mathcal{D}_1 \psi)_{\dots AB} = -(\mathcal{D}_1 \psi)_{\dots BA}$ . We have  $\mathcal{D}_1 \circ \mathcal{D}_0 = 0$  (cf. (2.11) in [6]) because  $\mathcal{D}_0$  and  $\mathcal{D}_1$  as differential operators are both densely-defined and closed, and for any  $u \in C^2(\Omega, \odot^k \mathbb{C}^2)$ ,

$$(2.19) \quad \begin{aligned} (\mathcal{D}_1 \mathcal{D}_0 u)_{A'_3 \dots A'_k AB} &= \frac{1}{2} \sum_{A'=0', 1'} \left( Z_A^{A'} (\mathcal{D}_0 u)_{A' A'_3 \dots A'_k B} - Z_B^{A'} (\mathcal{D}_0 u)_{A' A'_3 \dots A'_k A} \right) \\ &= \frac{1}{2} \sum_{A', C'=0', 1'} \left( Z_A^{A'} Z_B^{C'} u_{C' A' A'_3 \dots A'_k} - Z_B^{A'} Z_A^{C'} u_{C' A' A'_3 \dots A'_k} \right) = 0 \end{aligned}$$

by relabeling indices,  $u_{C' A' A'_3 \dots A'_k} = u_{A' C' A'_3 \dots A'_k}$  and the commutativity  $Z_B^{A'} Z_A^{C'} = Z_A^{C'} Z_B^{A'}$ , which holds for scalar differential operators of constant complex coefficients.

We have isomorphisms

$$(2.20) \quad \odot^k \mathbb{C}^2 \cong \mathbb{C}^{k+1}, \quad \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^{2k}, \quad \odot^{k-2} \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^2 \cong \mathbb{C}^{k-1}$$

by identification  $u \in \odot^k \mathbb{C}^2$ ,  $f \in \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2$ ,  $F \in \odot^{k-2} \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^2$  with

$$(2.21) \quad \begin{pmatrix} u_{0' \dots 0' 0'} \\ u_{1' \dots 0' 0'} \\ \vdots \\ u_{1' \dots 1' 0'} \\ u_{1' \dots 1' 1'} \end{pmatrix}, \quad \begin{pmatrix} f_{0' \dots 0' 0} \\ f_{0' \dots 0' 1} \\ \vdots \\ f_{1' \dots 1' 0} \\ f_{1' \dots 1' 1} \end{pmatrix}, \quad \begin{pmatrix} F_{0' \dots 0' 01} \\ F_{1' \dots 0' 01} \\ \vdots \\ F_{1' \dots 1' 01} \end{pmatrix},$$



respectively. Then the  $k$ -Cauchy-Fueter complexes become

$$0 \rightarrow C^\infty(\Omega, \mathbb{C}^{k+1}) \xrightarrow{\mathcal{D}_0} C^\infty(\Omega, \mathbb{C}^{2k}) \xrightarrow{\mathcal{D}_1} C^\infty(\Omega, \mathbb{C}^{k-1}) \rightarrow 0.$$

See also [6] for the matrix form of the Neumann problem (1.14) for general  $k$ . But we use a different norm of  $\odot^k \mathbb{C}^2$  there.

### 3. THE MODEL CASE $k = 2$

#### 3.1. The Neumann boundary condition.

**Proposition 3.1.** *For  $f \in C_0^1(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2)$ ,  $F \in C_0^1(\Omega, \wedge^2 \mathbb{C}^2)$ , we have*

$$(3.1) \quad (\mathcal{D}_0^* f)_{A'B'} = \sum_{A=0,1} \delta_{(A'}^A f_{B')A}, \quad (\mathcal{D}_1^* F)_{A'A} = \sum_{B=0,1} \delta_{A'}^B F_{BA}.$$

(2) Suppose that  $r$  is a defining function of  $\Omega$  with  $|dr| = 1$  on the boundary. Then  $f \in C^1(\overline{\Omega}, \mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$  if and only if

$$(3.2) \quad \sum_{A=0,1} Z_{(A'}^A r \cdot f_{B')A} = 0 \quad \text{on } \partial\Omega$$

for any  $A', B' = 0', 1'$ ; and  $F \in \text{Dom} \mathcal{D}_1^* \cap C^2(\overline{\Omega}, \wedge^2 \mathbb{C}^2)$  if and only if  $F = 0$  on  $\partial\Omega$ .

*Proof.* (1) For any  $u \in C_0^1(\Omega, \odot^2 \mathbb{C}^2)$ ,

$$\begin{aligned} \langle \mathcal{D}_0 u, f \rangle_\varphi &= \sum_{A,A',B'} \left( Z_A^{A'} u_{A'B'}, f_{B'A} \right)_\varphi = \sum_{A,A',B'} \left( u_{A'B'}, \delta_{A'}^A f_{B'A} \right)_\varphi \\ &= \sum_{A',B'} \left( u_{A'B'}, \sum_A \delta_{(A'}^A f_{B')A} \right)_\varphi = \langle u, \mathcal{D}_0^* f \rangle_\varphi \end{aligned}$$

by using  $\delta_{A'}^A$  in (2.6) as the formal adjoint operator of  $Z_A^{A'}$  and using symmetrisation by Lemma 2.1 (1). Here we have to symmetrise  $(A'B')$  in  $\sum_A \delta_{A'}^A f_{B'A}$  since only after symmetrisation it becomes a  $\odot^2 \mathbb{C}^2$ -valued function. Similarly,

$$\begin{aligned} \langle \mathcal{D}_1 h, F \rangle_\varphi &= \sum_{A,B,A'} \left( Z_{[A}^{A'} h_{B]A'}, F_{AB} \right)_\varphi = \sum_{A,B,A'} \left( Z_A^{A'} h_{BA'}, F_{AB} \right)_\varphi \\ &= \sum_{B,A'} \left( h_{BA'}, \sum_A \delta_{A'}^A F_{AB} \right)_\varphi = \langle h, \mathcal{D}_1^* F \rangle_\varphi \end{aligned}$$

for any  $h \in C_0^1(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2)$ , by dropping antisymmetrisation by Lemma 2.1 (2).

(2) For  $f \in C^2(\overline{\Omega}, \mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$  and any  $u \in C^1(\overline{\Omega}, \odot^2 \mathbb{C}^2)$  we have

$$(3.3) \quad \begin{aligned} \langle \mathcal{D}_0 u, f \rangle_\varphi &= \sum_{A,A',B'} \left( Z_A^{A'} u_{A'B'}, f_{B'A} \right)_\varphi = \sum_{A,A',B'} \left( u_{A'B'}, \delta_{A'}^A f_{B'A} \right)_\varphi + \mathcal{B} \\ &= \sum_{A',B'} \left( u_{A'B'}, \sum_A \delta_{(A'}^A f_{B')A} \right)_\varphi + \mathcal{B} \end{aligned}$$

by applying Stokes' formula (2.8) to the complex vector field  $Z_A^{A'}$  and using symmetrisation by Lemma 2.1 (1) (since  $u_{A'B'} = u_{B'A'}$ ), where the boundary term is

$$(3.4) \quad \mathcal{B} := - \int_{\partial\Omega} \sum_{A',B'} u_{A'B'} \sum_A \overline{Z_{A'}^A r \cdot f_{B'A}} e^{-\varphi} dS = - \int_{\partial\Omega} \sum_{A',B'} u_{A'B'} \sum_A \overline{Z_{(A'}^A r \cdot f_{B')A}} e^{-\varphi} dS.$$

If we choose  $u \in C_0^1(\Omega, \odot^2 \mathbb{C}^2)$  in (3.3), we see that  $\mathcal{B} = 0$  and consequently  $\mathcal{D}_0^* f$  coincides with the action of the formal adjoint operator (3.1) of  $\mathcal{D}_0$  on  $f$ . Thus  $\langle \mathcal{D}_0 u, f \rangle_\varphi = \langle u, \mathcal{D}_0^* f \rangle_\varphi$  for any  $u \in \text{Dom}(\mathcal{D}_0)$  if and only if the boundary term  $\mathcal{B}$  vanishes, i.e  $f \in \text{Dom}(\mathcal{D}_0^*)$  if and only if (3.2) holds. Here we have to symmetrise (3.4), since  $(u_{A'B'})$  have only 3 independent components:  $u_{0'0'}$ ,  $u_{0'1'} = u_{1'0'}$  and  $u_{1'1'}$ .

Now for  $h \in C^2(\overline{\Omega}, \mathbb{C}^2 \otimes \mathbb{C}^2)$  and  $F \in C^1(\overline{\Omega}, \Lambda^2 \mathbb{C}^2)$ , we see that

$$\begin{aligned} \langle \mathcal{D}_1 h, F \rangle_\varphi &= \sum_{B,A,A'} \left( Z_{[B}^{A'} h_{A]A'}, F_{BA} \right)_\varphi = \sum_{B,A,A'} \left( Z_B^{A'} h_{AA'}, F_{BA} \right)_\varphi \\ &= \sum_{A,A'} \left( h_{AA'}, \sum_B \delta_{A'}^B F_{BA} \right)_\varphi + \mathcal{B}', \end{aligned}$$

by dropping antisymmetrisation by Lemma 2.1 (2) and applying Stokes' formula (2.8) again. So  $\langle \mathcal{D}_1 h, F \rangle_\varphi = \langle h, \mathcal{D}_1^* F \rangle_\varphi$  holds if and only if the boundary term  $\mathcal{B}' := - \int_{\partial\Omega} \sum_{AA'} h_{AA'} \overline{Z_{A'}^B r} \cdot \overline{F_{BA}} e^{-\varphi} dS = 0$ . Namely  $\sum_B Z_{A'}^B r \cdot F_{BA} = 0$  on  $\partial\Omega$  for any  $A' = 0', 1', A = 0, 1$ . Thus  $Z_0^A r \cdot F_{01} = 0$  for  $A = 0, 1$ . Note that  $\sum_{A'} |Z_0^{A'} r|^2 = |dr|^2 = 1$  by (4.14). It follows that  $F_{01} = 0$  on  $\partial\Omega$ .  $\square$

### 3.2. 2-plurisubharmonicity.

**Proposition 3.2.** *Suppose that  $\chi$  is an increasing smooth convex function over  $[0, \infty)$ . Then for  $\psi(x) = \chi(\varphi(x))$ , we have*

$$(3.5) \quad \mathcal{L}_2(\psi; \xi) \geq \chi'(\varphi) \mathcal{L}_2(\varphi; \xi).$$

*In particular  $\psi$  is 2-plurisubharmonic if  $\varphi$  is.*

*Proof.* It is direct to see that for any  $(\xi_{A'A}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ ,

$$\begin{aligned} \mathcal{L}_2(\psi; \xi)(x) &= - \sum_{A,B,A',B'} \left( Z_B^{A'} Z_{A'}^A \psi(x) \cdot \xi_{B'A} + Z_B^{A'} Z_{B'}^A \psi(x) \cdot \xi_{A'A} \right) \overline{\xi_{B'B}} \\ &= - \chi'(\varphi(x)) \sum_{A,B,A',B'} \left( Z_B^{A'} Z_{A'}^A \varphi(x) \cdot \xi_{B'A} + Z_B^{A'} Z_{B'}^A \varphi(x) \cdot \xi_{A'A} \right) \overline{\xi_{B'B}} \\ &\quad - \chi''(\varphi(x)) \sum_{A,B,A',B'} Z_B^{A'} \varphi(x) \left( Z_{A'}^A \varphi(x) \cdot \xi_{B'A} + Z_{B'}^A \varphi(x) \cdot \xi_{A'A} \right) \overline{\xi_{B'B}} \\ &= \chi'(\varphi(x)) \mathcal{L}_2(\varphi; \xi)(x) + 2\chi''(\varphi(x)) \sum_{A',B'} \sum_B \overline{Z_{A'}^B \varphi(x) \xi_{B'B}} \cdot \sum_A Z_{(A')}^A \varphi(x) \xi_{B'A} \\ &= \chi'(\varphi(x)) \mathcal{L}_2(\varphi; \xi)(x) + 2\chi''(\varphi(x)) \sum_{A',B'} \left| \sum_A Z_{(A')}^A \varphi(x) \xi_{B'A} \right|^2 \geq \chi'(\varphi(x)) \mathcal{L}_2(\varphi; \xi)(x) \end{aligned}$$

by  $-Z_B^{A'} \varphi = \overline{Z_{A'}^B \varphi}$  for real  $\varphi$  by (2.3) and using symmetrisation by Lemma 2.1 (1). The result follows  $\square$

2-plurisubharmonic functions and 2-pseudoconvex domains are abundant by the following examples, and by definition a small perturbation of a strictly 2-plurisubharmonic function is still strictly 2-plurisubharmonic.

**Proposition 3.3.**  *$r_1(x) = x_1^2 + x_2^2$  and  $r_2(x) = x_3^2 + x_4^2$  are both strictly 2-plurisubharmonic, and  $\chi_1(r_1)$ ,  $\chi_2(r_2)$  and their sum are all (strictly) 2-plurisubharmonic for any increasing smooth (strictly) convex functions  $\chi_1$  and  $\chi_2$  over  $[0, \infty)$ .*

*Proof.* Note that

$$(3.6) \quad \mathcal{L}_2(\psi; \xi) = \sum_{A, B, A', B'} \left( Z_B^{A'} \overline{Z_A^{A'}} \psi \cdot \xi_{B'A} \overline{\xi_{B'B}} + Z_B^{A'} \overline{Z_A^{B'}} \psi \cdot \xi_{A'A} \overline{\xi_{B'B}} \right)$$

by definition of  $\mathcal{L}_2$  and (2.3), and

$$(3.7) \quad (Z_A^{A'}) = \begin{pmatrix} -\partial_{x_3} - \mathbf{i}\partial_{x_4} & -\partial_{x_1} - \mathbf{i}\partial_{x_2} \\ \partial_{x_1} - \mathbf{i}\partial_{x_2} & -\partial_{x_3} + \mathbf{i}\partial_{x_4} \end{pmatrix},$$

by (1.4) and (2.4). So  $Z_0^{0'}$  and  $Z_1^{1'}$  are independent of  $x_0$  and  $x_1$ . According to  $(A, B) = (0, 0), (1, 1), (1, 0)$  and  $(0, 1)$  in (3.6), we get

$$\begin{aligned} \mathcal{L}_2(r_1; \xi) &= \sum_{B'} Z_0^{1'} \overline{Z_0^{1'}} r_1 \cdot \xi_{B'0} \overline{\xi_{B'0}} + Z_0^{1'} \overline{Z_0^{1'}} r_1 \cdot \xi_{1'0} \overline{\xi_{1'0}} \\ &\quad + \sum_{B'} Z_1^{0'} \overline{Z_1^{0'}} r_1 \cdot \xi_{B'1} \overline{\xi_{B'1}} + Z_1^{0'} \overline{Z_1^{0'}} r_1 \cdot \xi_{0'1} \overline{\xi_{0'1}} + Z_0^{1'} \overline{Z_1^{0'}} r_1 \cdot \xi_{1'1} \overline{\xi_{0'0}} + Z_1^{0'} \overline{Z_0^{1'}} r_1 \cdot \xi_{0'0} \overline{\xi_{1'1}} \\ &= 4 \sum_{B', A} |\xi_{B'A}|^2 + 4(|\xi_{1'0}|^2 + |\xi_{0'1}|^2) \geq 4|\xi|^2 \end{aligned}$$

by

$$(3.8) \quad Z_0^{1'} \overline{Z_0^{1'}} r_1 = 4 = Z_0^{1'} \overline{Z_1^{0'}} r_1 \quad \text{and} \quad Z_0^{1'} \overline{Z_1^{0'}} r_1 = 0 = Z_0^{1'} \overline{Z_1^{0'}} r_1$$

since

$$(3.9) \quad Z_0^{1'} \overline{Z_0^{1'}} = \partial_{x_1}^2 + \partial_{x_2}^2 = Z_1^{0'} \overline{Z_1^{0'}} \quad \text{and} \quad Z_0^{1'} \overline{Z_1^{0'}} = -\partial_{x_1}^2 + \partial_{x_2}^2 - 2\mathbf{i}\partial_{x_1}\partial_{x_2}.$$

Similarly, we have  $\mathcal{L}_2(r_2; \xi) \geq 4|\xi|^2$  and so  $\mathcal{L}_2(\chi_1(r_1) + \chi_2(r_2); \xi) \geq 4(\chi_1'(r_1) + \chi_2'(r_2))|\xi|^2$  by (3.5).  $\square$

**3.3. 2-pseudoconvexity.** For a 2-pseudoconvex domain, we can choose a defining function satisfying  $|dr| = 1$  on the boundary by the following proposition,

**Proposition 3.4.** *2-pseudoconvexity of a domain is independent of the choice of the defining function.*

*Proof.* Suppose  $r$  and  $\tilde{r}$  are both defining functions of the domain  $\Omega$ . Then  $\tilde{r}(x) = \mu(x)r(x)$  for some nonvanishing function  $\mu > 0$  near  $\partial\Omega$ . Note that  $Z_{A'}^A \tilde{r} = \mu \cdot Z_{A'}^A r$  on the boundary. It is obvious that for any  $A', B' = 0', 1'$ ,  $(\xi_{A'A}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$  satisfies  $\sum_{A=0,1} Z_{(A', \tilde{r})}^A \cdot \xi_{B'A} = 0$  on  $\partial\Omega$  if and only if

$$\sum_{A=0,1} Z_{(A', r)}^A \cdot \xi_{B'A} = 0, \quad \text{on } \partial\Omega.$$

So the boundary condition for  $\xi$  is independent of the choice of the defining function. Then for fixed  $x \in \partial\Omega$ , we have

$$\begin{aligned} \mathcal{L}_2(\tilde{r}; \xi)(x) &= - \sum_{A, B, A', B'} \left\{ Z_B^{A'} Z_{A'}^A (\mu r)(x) \xi_{B'A} + Z_B^{A'} Z_{B'}^A (\mu r)(x) \cdot \xi_{A'A} \right\} \overline{\xi_{B'B}} \\ &= \mu(x) \mathcal{L}_2(r; \xi)(x) + r(x) \mathcal{L}_2(\mu; \xi)(x) \\ &\quad - \sum_{A, B, A', B'} \left\{ Z_B^{A'} \mu(x) Z_{A'}^A r(x) + Z_B^{A'} r(x) Z_{A'}^A \mu(x) \right\} \xi_{B'A} \overline{\xi_{B'B}} \\ &\quad - \sum_{A, B, A', B'} \left\{ Z_B^{A'} \mu(x) Z_{B'}^A r(x) + Z_B^{A'} r(x) Z_{B'}^A \mu(x) \right\} \cdot \xi_{A'A} \overline{\xi_{B'B}} \\ &= : \mu(x) \mathcal{L}_2(r; \xi) + 0 + \Sigma, \end{aligned}$$

by direct calculation and  $r|_{\partial\Omega} = 0$ . We use  $Z_B^{A'} = -\overline{Z_{A'}^B}$  in (2.3) to rewrite  $\Sigma$  as

$$\begin{aligned}\Sigma &= \sum_{A', B'} \left\{ - \sum_B Z_B^{A'} \mu(x) \overline{\xi_{B'B}} \left( \sum_A Z_{A'}^A r(x) \xi_{B'A} + \sum_A Z_{B'}^A r(x) \xi_{A'A} \right) \right. \\ &\quad \left. + \sum_B \overline{Z_{A'}^B r(x) \xi_{B'B}} \left( \sum_A Z_{A'}^A \mu(x) \xi_{B'A} + \sum_A Z_{B'}^A \mu(x) \xi_{A'A} \right) \right\} \\ &= 2 \sum_{A', B'} \sum_B \overline{Z_{A'}^B r(x) \xi_{B'B}} \sum_A Z_{(A')\mu(x) \xi_{B'A}}^A \\ &= 2 \sum_{A', B'} \sum_B \overline{Z_{(A')r(x) \xi_{B'B}}^B} \cdot \sum_A Z_{(A')\mu(x) \xi_{B'A}}^A = 0\end{aligned}$$

for  $\xi$  satisfying the boundary condition (1.11), by using symmetrisation by Lemma 2.1 (1) and the boundary condition (1.11) for  $\xi$  twice. At last we get  $\mathcal{L}_2(\tilde{r}; \xi) = \mu(x) \mathcal{L}_2(r; \xi)$  on the boundary for  $\xi$  satisfying the boundary condition (1.11). The result follows.  $\square$

**Remark 3.1.** The 2-pseudoconvexity in (1.10) is the natural convexity associated to the 2-Cauchy-Fueter complex (cf. Hörmander [13] for the notion of convexity associated to a differential operator). The 2-pseudoconvexity similarly defined in  $\mathbb{R}^{4n}$  is different from the pseudoconvexity introduced in [22], which is based on the notion of a plurisubharmonic function over quaternionic space introduced by Alesker [1].

#### 4. THE $L^2$ -ESTIMATE IN THE MODEL CASE $k = 2$

**4.1. The  $L^2$ -estimate for the case  $k = 2$ .** By the following density Lemma 4.1, it is sufficient to show the  $L^2$  estimate (1.17) for  $f \in C^\infty(\overline{\Omega}, \mathbb{C}^2 \otimes \mathbb{C}^2)$  satisfying the Neumann boundary condition (1.11). By definition and expanding symmetrisation, we get

$$\begin{aligned}(4.1) \quad 2\langle \mathcal{D}_0^* f, \mathcal{D}_0^* f \rangle_\varphi &= 2\langle \mathcal{D}_0 \mathcal{D}_0^* f, f \rangle_\varphi = 2 \sum_{B, B'} \left( \sum_{A'} Z_B^{A'} \sum_A \delta_{(A')f_{B'A}}^A, f_{B'B} \right)_\varphi \\ &= \sum_{A, B, A', B'} \left( Z_B^{A'} \delta_{A'}^A f_{B'A} + Z_B^{A'} \delta_{B'}^A f_{A'A}, f_{B'B} \right)_\varphi.\end{aligned}$$

Using commutators

$$(4.2) \quad [Z_B^{A'}, \delta_{B'}^A] = -Z_B^{A'} Z_{B'}^A \varphi$$

for any fixed  $A, A', B, B'$  by  $Z_A^{A'}$ 's mutually commutative (since they are of constant coefficients), we get

$$\begin{aligned}(4.3) \quad 2\|\mathcal{D}_0^* f\|_\varphi^2 &= \sum_{A, B, A', B'} \left\{ \left( \delta_{A'}^A Z_B^{A'} f_{B'A} + \delta_{B'}^A Z_B^{A'} f_{A'A}, f_{B'B} \right)_\varphi \right. \\ &\quad \left. + \left( [Z_B^{A'}, \delta_{A'}^A] f_{B'A} + [Z_B^{A'}, \delta_{B'}^A] f_{A'A}, f_{B'B} \right)_\varphi \right\} \\ &= \sum_{A, B, A', B'} \left( Z_B^{A'} f_{B'A}, Z_{A'}^A f_{B'B} \right)_\varphi + \sum_{A, B, A', B'} \left( Z_B^{A'} f_{A'A}, Z_{A'}^B f_{B'B} \right)_\varphi + \mathcal{B}_2 \\ &\quad - \sum_{A, B, A', B'} \left( Z_B^{A'} Z_{A'}^A \varphi \cdot f_{B'A} + Z_B^{A'} Z_{B'}^A \varphi \cdot f_{A'A}, f_{B'B} \right)_\varphi \\ &=: \Sigma_1 + \Sigma_2 + \mathcal{B}_2 + \int_\Omega \mathcal{L}_2(\varphi; f(x)) e^{-\varphi(x)} dV,\end{aligned}$$

by using Stokes' formula (2.9), where  $\mathcal{B}_2$  is the boundary term

$$\mathcal{B}_2 := \int_{\partial\Omega} \sum_{A,B,A',B'} \left( Z_{A'}^A r \cdot Z_B^{A'} f_{B'A} + Z_{B'}^A r \cdot Z_B^{A'} f_{A'A} \right) \cdot \overline{f_{B'B}} e^{-\varphi} dS,$$

which looks more complicated than the boundary term appearing in the  $\bar{\partial}$ -Neumann problem in several complex variables. But fortunately it can be handled similarly by Morrey's technique for solving the  $\bar{\partial}$ -Neumann problem [7] as follows. Since  $\sum_{A=1}^2 Z_{(A',r}^A f_{B')A}$  vanishes on the boundary by the boundary condition (1.13) for fixed  $A', B' = 0', 1'$ , there exists 3 functions  $\lambda_{A'B'}$  ( $\lambda_{A'B'} = \lambda_{B'A'}$ ) such that

$$(4.4) \quad \sum_{A=0,1} \{ Z_{A'}^A r \cdot f_{B'A} + Z_{B'}^A r \cdot f_{A'A} \} = \lambda_{A'B'} \cdot r,$$

near the boundary. Now differentiate the equation (4.4) by the complex vector field  $Z_B^{A'}$  to get

$$\sum_A \left\{ Z_B^{A'} Z_{A'}^A r \cdot f_{B'A} + Z_B^{A'} Z_{B'}^A r \cdot f_{A'A} + Z_{A'}^A r \cdot Z_B^{A'} f_{B'A} + Z_{B'}^A r \cdot Z_B^{A'} f_{A'A} \right\} = Z_B^{A'} \lambda_{A'B'} \cdot r + \lambda_{A'B'} \cdot Z_B^{A'} r.$$

Then multiplying it by  $\overline{f_{B'B}}$  and taking summation over  $B, A', B'$ , we get that

$$\begin{aligned} \sum_{A,B,A',B'} \left\{ 2Z_B^{A'} Z_{(A',r}^A f_{B')A} + Z_{A'}^A r \cdot Z_B^{A'} f_{B'A} + Z_{B'}^A r \cdot Z_B^{A'} f_{A'A} \right\} \overline{f_{B'B}} &= - \sum_{B,A',B'} \lambda_{A'B'} \cdot \overline{Z_{A'}^B r \cdot f_{B'B}} \\ &= - \sum_{A',B'} \left( \lambda_{A'B'} \sum_B \overline{Z_{(A',r}^B f_{B')B}} \right) = 0 \end{aligned}$$

on the boundary  $\partial\Omega$  by  $\lambda$  symmetric in the primed indices and the boundary condition (1.13) for  $f$ . Thus it follows that the boundary term becomes

$$(4.5) \quad \mathcal{B}_2 = - \int_{\partial\Omega} 2 \sum_{A,B,A',B'} Z_B^{A'} Z_{(A',r}^A f_{B')A} \cdot \overline{f_{B'B}} e^{-\varphi} dS = \int_{\partial\Omega} \mathcal{L}_2(r; f(x)) e^{-\varphi(x)} dS \geq 0$$

for  $f$  satisfying the boundary condition (1.13) by the pseudoconvexity (1.10)-(1.11) of  $r$ .

For the second sum in (4.3), we have

$$\begin{aligned} (4.6) \quad \Sigma_2 &= \sum_{A,B} \left( \sum_{A'} Z_B^{A'} f_{AA'}, \sum_{B'} Z_A^{B'} f_{BB'} \right)_{\varphi} = \sum_{A,B} \left( \left\| \sum_{A'} Z_A^{A'} f_{BA'} \right\|_{\varphi}^2 - 2 \left\| \sum_{A'} Z_{[A}^{A'} f_{B]A'} \right\|_{\varphi}^2 \right) \\ &= \sum_{A,B} \left\| \sum_{A'} Z_A^{A'} f_{BA'} \right\|_{\varphi}^2 - 2 \|\mathcal{D}_1 f\|_{\varphi}^2 \end{aligned}$$

by applying Lemma 2.1 (3) with  $h_{BA} = \sum_{A'} Z_B^{A'} f_{AA'}$ ,  $H_{AB} = \sum_{B'} Z_A^{B'} f_{BB'}$  and using definition (2.18) of  $\mathcal{D}_1 f$ .

For the first sum in (4.3), we have

$$(4.7) \quad \Sigma_1 = \sum_{A',B'} \sum_{A,B} \left( Z_B^{A'} f_{AB'}, Z_A^{A'} f_{BB'} \right)_{\varphi} = \sum_{A',B'} \sum_{A,B} \left( \left\| Z_A^{A'} f_{BB'} \right\|_{\varphi}^2 - 2 \left\| Z_{[A}^{A'} f_{B]B'} \right\|_{\varphi}^2 \right)$$

by applying Lemma 2.1 (3) again with  $h_{BA} = Z_B^{A'} f_{AB'}$ ,  $H_{AB} = Z_A^{A'} f_{BB'}$  for fixed  $A', B'$ . In general  $\Sigma_1$  may be negative. To control the last term in the right hand side, note that it does not vanish only when  $[AB] = [01]$ , and

$$(4.8) \quad Z_A^{0'} = Z_{1'A} \quad \text{and} \quad Z_A^{1'} = -Z_{0'A},$$

by (2.4). So we can relabel indices to get

$$(4.9) \quad \sum_{A,B,A',B'} \left\| Z_{[A}^{A'} f_{B]}_{B'} \right\|_{\varphi}^2 = 2 \sum_{A',B'} \left\| Z_{A'[0]f_1]_{B'}} \right\|_{\varphi}^2.$$

Now substituting (4.5)-(4.7) and (4.9) into (4.3), we get the estimate

$$(4.10) \quad 2\|\mathcal{D}_0^* f\|_{\varphi}^2 + 2\|\mathcal{D}_1 f\|_{\varphi}^2 \geq \int_{\Omega} \mathcal{L}_2(\varphi; f) e^{-\varphi} dV - 4 \sum_{A',B'} \left\| Z_{A'[0]f_1]_{B'}} \right\|_{\varphi}^2.$$

To control the last term, note that for fixed  $A', B'$ ,

$$(4.11) \quad \begin{aligned} 2Z_{A'[0]f_1]_{B'}} &= Z_{A'0}f_{1B'} - Z_{A'1}f_{0B'} = - \sum_{A=0,1} Z_{A'}^A f_{B'A} \\ &= - \sum_A Z_{(A'}^A f_{B')A} - \sum_A Z_{[A'}^A f_{B']A} \end{aligned}$$

by using (2.13) and the identity

$$u_{A'B'} = \frac{1}{2}(u_{A'B'} + u_{B'A'}) + \frac{1}{2}(u_{A'B'} - u_{B'A'}) = u_{(A'B')} + u_{[A'B']}.$$

It follows that

$$(4.12) \quad 4 \sum_{A',B'} \left\| Z_{A'[0]f_1]_{B'}} \right\|_{\varphi}^2 \leq \sum_{A',B'} \left\| \sum_A Z_{(A'}^A f_{B')A} \right\|_{\varphi}^2 + \sum_{A',B'} \left\| \sum_A Z_{[A'}^A f_{B']A} \right\|_{\varphi}^2 =: \Sigma'_1 + \Sigma''_1.$$

Note that similar to (4.11), we have

$$\begin{aligned} \sum_{A=0,1} Z_{[0'}^A f_{1']A} &= \frac{1}{2} \sum_{A=0,1} (Z_{0'}^A f_{A1'} - Z_{1'}^A f_{A0'}) = - \sum_{A=0,1} (Z_{0'[0]f_1]_{1'}} - Z_{1'[0]f_1]_{0'}}) \\ &= \sum_{A'=0',1'} Z_{[0'}^{A'} f_{1']_{A'}} = (\mathcal{D}_1 f)_{01}, \end{aligned}$$

by using (2.13) for primed and unprimed indices, respectively. Thus  $\Sigma''_1 \leq \|\mathcal{D}_1 f\|_{\varphi}^2$ . On the other hand,  $\Sigma'_1$  in (4.12) can be simply controlled by  $\|\mathcal{D}_0^* f\|_{\varphi}^2$  in the following way:

$$(4.13) \quad \begin{aligned} \Sigma'_1 &= \sum_{A',B'} \left\| \sum_A \delta_{(A'}^A f_{B')A} + \sum_A Z_{(A'}^A \varphi \cdot f_{B')A} \right\|_{\varphi}^2 \\ &\leq 2 \sum_{A',B'} \left\| \sum_A \delta_{(A'}^A f_{B')A} \right\|_{\varphi}^2 + 2 \sum_{A',B'} \left\| \left( \sum_A |Z_{A'}^A \varphi|^2 \right)^{\frac{1}{2}} \left( \sum_A |f_{AB'}|^2 \right)^{\frac{1}{2}} \right\|_{\varphi}^2 \\ &\leq 2\|\mathcal{D}_0^* f\|_{\varphi}^2 + 4\|d\varphi\|_{\infty}^2 \cdot \|f\|_{\varphi}^2, \end{aligned}$$

by  $|a+b|^2 \leq 2(|a|^2 + |b|^2)$  and the Cauchy-Schwarz inequality, where

$$(4.14) \quad |d\varphi|^2 = \sum_A |Z_{0'}^A \varphi|^2 = \sum_A |Z_{1'}^A \varphi|^2$$

by  $\varphi$  real and

$$(4.15) \quad (Z_{A'}^A) = \begin{pmatrix} \partial_{x_3} - \mathbf{i}\partial_{x_4} & -\partial_{x_1} - \mathbf{i}\partial_{x_2} \\ \partial_{x_1} - \mathbf{i}\partial_{x_2} & \partial_{x_3} + \mathbf{i}\partial_{x_4} \end{pmatrix},$$

by definition of  $Z_{A'}^A$  in (2.5) and (1.4). Substitute these estimates to (4.12) to get

$$4 \sum_{A', B'} \|Z_{A'}[0f_1]_{B'}\|_\varphi^2 \leq 2\|\mathcal{D}_0^* f\|_\varphi^2 + 4\|d\varphi\|_\infty^2 \cdot \|f\|_\varphi^2 + \|\mathcal{D}_1 f\|_\varphi^2.$$

Now substitute this into (4.10) to get

$$(4.16) \quad 4\|\mathcal{D}_0^* f\|_\varphi^2 + 4\|\mathcal{D}_1 f\|_\varphi^2 \geq \int_\Omega \mathcal{L}_2(\varphi; f) e^{-\varphi} dV - 4\|d\varphi\|_\infty^2 \cdot \|f\|_\varphi^2.$$

Here  $\mathcal{L}_2(\varphi; f(x)) \geq c|f(x)|^2$ . The  $L^2$ -estimate (1.17) for  $k = 2$  follows.  $\square$

**Remark 4.1.** (1) For the weight  $\kappa\varphi$ , we have  $\mathcal{L}_2(\kappa\varphi; \xi) = \kappa\mathcal{L}_2(\varphi; \xi) \geq c\kappa|\xi|^2$  for  $\xi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ , and  $c\kappa - 4\|d(\kappa\varphi)\|_\infty^2 = \kappa(c - 4\kappa\|d\varphi\|_\infty^2) > 0$  if  $0 < \kappa < \frac{c}{4\|d\varphi\|_\infty^2}$ .

(2) On  $\mathbb{R}^{4n}$  with  $n > 1$ , the term in (4.9) becomes  $\sum_{A, B=0}^{2n-1} \sum_{A', B'} \|Z_{[A}^{A'} f_{B]B'}\|_\varphi^2$ , which can not be simply estimated by using (2.13). But over the whole space  $\mathbb{R}^{4n}$  with weight  $\varphi = |x|^2$ , if we do not handle the nonnegative term  $\sum_{A, B, A', B'} (Z_B^{A'} \delta_{B'}^A f_{A'A}, f_{B'B})_\varphi$  in (4.1) by using commutators and drop it directly, we can obtain a weak  $L^2$  estimate (cf. [23]).

**4.2. The Density Lemma.** The following density lemma can be deduced from a general result due to Hörmander [13] (see also [18]).

**Lemma 4.1.**  $C^\infty(\overline{\Omega}, \mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$  is dense in  $\text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$ .

In general, let  $\Omega \subset \mathbb{R}^N$  be an open set and let

$$(4.17) \quad L(x, \partial) = \sum_{j=1}^N L_j(x) \partial_{x_j} + L_0(x)$$

be a first-order differential operator, where  $L_j \in C^1(\Omega, \mathbb{C}^{K \times J})$ ,  $j = 0, 1, \dots, N$ . Here  $\mathbb{C}^{K \times J}$  denotes the space of  $K \times J$  matrices with complex coefficients. The graph of the *maximal differential operator* defined by  $L(x, \partial)$  in  $L^2$  consists of all pairs  $(f, u) \in L^2(\Omega, \mathbb{C}^J) \times L^2(\Omega, \mathbb{C}^K)$  such that  $L(x, \partial)f = u$  in the sense of distributions, i.e.

$$(4.18) \quad \langle f, L^*(x, \partial)v \rangle = \langle u, v \rangle, \quad \text{for any } v \in C_0^\infty(\Omega, \mathbb{C}^K)$$

where  $L^*(x, \partial) = \sum L_j^*(x) \partial_{x_j} + L_0^*(x) + \sum \partial_{x_j} L_j^*(x)$  is the formal adjoint operator of  $L(x, \partial)$ . Then the maximal differential operator defined by  $L(x, \partial)$  is the adjoint of the differential operator  $L^*(x, \partial)$  with domain  $C_0^\infty(\Omega, \mathbb{C}^K)$ . So the maximal operator is closed and its adjoint is the closure of  $L^*(x, \partial)$ , first defined with domain  $C_0^\infty(\Omega, \mathbb{C}^K)$ . It is called the *minimal operator* defined by  $L^*(x, \partial)$ . Similarly, the adjoint of the maximal operator defined by  $L^*(x, \partial)$  is the minimal operator defined by  $L(x, \partial)$ .

**Proposition 4.1.** (proposition A.1 of [13]) If  $\Omega$  is a bounded domain with  $C^1$  boundary,  $L_j \in C^1$  and  $L_0 \in C^0$  in a neighborhood of  $\overline{\Omega}$ , then the maximal operator defined by  $L(x, \partial)$  in (4.17) is the closure of its restriction to functions which are  $C^\infty$  in a neighborhood of  $\overline{\Omega}$ . The minimal domain of  $L(x, \partial)$  consists precisely of the functions  $f \in L^2(\Omega, \mathbb{C}^J)$  such that  $L(x, \partial)\tilde{f} \in L^2(\mathbb{R}^N, \mathbb{C}^K)$  if  $\tilde{f} := f$  in  $\Omega$  and  $\tilde{f} := 0$  in  $\Omega^c$ .

Consider another first-order differential operator  $M(x, \partial) = \sum M_j(x) \partial_{x_j} + M_0(x)$  where  $M_j \in C^1(U, \mathbb{C}^{K' \times J})$ ,  $j = 0, \dots, N$ . For an open neighborhood  $U$  of  $0 \in \mathbb{R}^N$ , denote  $U_- := \{x \in U; x_N < 0\}$ .

**Proposition 4.2.** (proposition A.2 of [13]) Assume that  $\ker L_N(x)$  and  $\ker L_N(x) \cap \ker M_N(x)$  have constant dimension when  $x \in U$ . If  $u \in L^2(U_-, \mathbb{C}^J)$  is in the minimal domain of  $L(x, \partial)$  and the maximal domain of  $M(x, \partial)$  in  $U_-$ , and if  $\text{supp } f$  is sufficiently close to the origin, then there exists a sequence  $f_\nu \in C_0^\infty(U)$  such that  $f_\nu$  restricted to  $U_-$  is in the minimal domain of  $L(x, \partial)$  and  $f_\nu \rightarrow f$ ,  $L(x, \partial)f_\nu \rightarrow L(x, \partial)f$ ,  $M(x, \partial)f_\nu \rightarrow M(x, \partial)f$  in  $L^2(U_-)$ .

*Proof of Lemma 4.1.* We denote by  $\mathcal{L}$  the differential operator given by the formal adjoint operator (3.1) of  $\mathcal{D}_0$ . If we use notations in (2.21) to identify linear spaces in (2.20) for  $k = 2$ ,  $\mathcal{D}_0$ ,  $\mathcal{D}_1$  and  $\mathcal{L}$  are  $4 \times 3$ -,  $1 \times 4$ - and  $3 \times 4$ -matrix valued differential operators of first order, respectively. In particular,

$$(4.19) \quad \mathcal{L}f = [\widehat{\mathcal{L}} - \widehat{\mathcal{L}}\varphi]f, \quad \text{with} \quad \widehat{\mathcal{L}} = \begin{pmatrix} Z_{0'}^0 & Z_{0'}^1 & 0 & 0 \\ \frac{1}{2}Z_{1'}^0 & \frac{1}{2}Z_{1'}^1 & \frac{1}{2}Z_{0'}^0 & \frac{1}{2}Z_{0'}^1 \\ 0 & 0 & Z_{1'}^0 & Z_{1'}^1 \end{pmatrix}, \quad f = \begin{pmatrix} f_{0'0} \\ f_{0'1} \\ f_{1'0} \\ f_{1'1} \end{pmatrix},$$

and the operator  $\mathcal{M} := \mathcal{D}_1$  takes the form

$$(4.20) \quad \mathcal{M}f = \frac{1}{2}(-Z_{1'}^0, Z_{0'}^0, -Z_{1'}^1, Z_{0'}^1) \begin{pmatrix} f_{0'0} \\ f_{0'1} \\ f_{1'0} \\ f_{1'1} \end{pmatrix}.$$

Since  $Z_A^{A'}$ 's and  $Z_{A'}^A$ 's are complex vector fields with constant coefficients, we can write

$$(4.21) \quad \mathcal{L} = \sum_{j=1}^4 \mathcal{L}_j \frac{\partial}{\partial x_j} + \mathcal{L}_0, \quad \mathcal{L}_0 = -\widehat{\mathcal{L}}\varphi, \quad \mathcal{M} = \sum_{j=1}^4 \mathcal{M}_j \frac{\partial}{\partial x_j},$$

where  $\mathcal{L}_j$ 's are constant  $3 \times 4$ -matrices and  $\mathcal{M}_j$ 's are constant  $1 \times 4$ -matrices.

By definition,  $\text{Dom}(\mathcal{D}_j)$  is exactly the domain of the maximal operator defined by  $\mathcal{D}_j$ ,  $j = 1, 2$ . To apply the above propositions, let us show that  $\text{Dom}(\mathcal{D}_0^*)$  coincides with the domain of the minimal operator defined by  $\mathcal{L}$ .  $f \in \text{Dom}(\mathcal{D}_0^*)$  if and only if there exists some  $u \in L^2(\Omega, \mathbb{C}^3)$  such that  $\langle \mathcal{D}_0 v, f \rangle_\varphi = \langle v, u \rangle_\varphi$  for any  $v \in \text{Dom}(\mathcal{D}_0)$ . In terms of the unweighted inner product  $\langle \cdot, \cdot \rangle$  in (4.18), it is equivalent to

$$\langle (\mathcal{D}_0 + \mathcal{D}_0\varphi)v, f \rangle = \langle v, u \rangle$$

for any  $v \in \text{Dom}(\mathcal{D}_0)$ , since  $\varphi$  is smooth on the bounded domain  $\Omega$ . Note that the minimal operator defined by  $\mathcal{L}$  is the adjoint operator of the maximal operator defined by  $\mathcal{D}_0 + \mathcal{D}_0\varphi$  with respect to the unweighted inner product (4.18). By abuse of notations, we denote the minimal operator defined by  $\mathcal{L}$  also by  $\mathcal{L}$  and the maximal operator defined by  $\mathcal{M}$  also by  $\mathcal{M}$ . so  $f$  is in the domain of the minimal operator defined by  $\mathcal{L}$ . The converse is also true.

Suppose that  $f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$ . Write  $\mathcal{L}f = u$ . By Proposition 4.1,  $f$  in the minimal domain of  $\mathcal{L}$  implies that

$$\mathcal{L}\tilde{f} = \tilde{u}$$

as  $L^2(\mathbb{R}^4, \mathbb{C}^3)$  functions. Suppose that  $\{\rho_\nu\}$  is a unit partition subordinated to a finite covering  $\{\mathcal{U}_\nu\}$  of  $\overline{\Omega}$  such that either  $\overline{\mathcal{U}_\nu} \subset \Omega$  or  $\mathcal{U}_\nu \cap \partial\Omega \neq \emptyset$ . Let  $I'$  and  $I''$  be the sets of corresponding indices  $\nu$ , respectively. Write  $f_\nu := \rho_\nu f$ . Then  $\sum_{\nu \in I' \cup I''} f_\nu = f$ , and

$$(4.22) \quad \mathcal{L}f_\nu = u_\nu \quad \text{with} \quad u_\nu = \rho_\nu \mathcal{L}f + \mathcal{L}\rho_\nu \cdot f.$$



Then  $\widetilde{\rho_\nu f} = \rho_\nu \widetilde{f}$  by definition, and so  $\mathcal{L}\widetilde{f}_\nu = \rho_\nu \mathcal{L}\widetilde{f} + \mathcal{L}\rho_\nu \widetilde{f} = \rho_\nu \widetilde{u} + \mathcal{L}\rho_\nu \widetilde{f} = \widetilde{u}_\nu$ . Hence

$$(4.23) \quad \mathcal{L}\widetilde{f}_\nu = \widetilde{\mathcal{L}f_\nu}.$$

For  $\nu \in I'$ , by Friderich's lemma, there exists a sequence  $f_{\nu;n} \in C_0^\infty(\mathcal{U}_\nu, \mathbb{C}^4)$  such that  $f_{\nu;n} \rightarrow f_\nu$ ,  $\mathcal{L}f_{\nu;n} \rightarrow \mathcal{L}f_\nu$ ,  $\mathcal{M}f_{\nu;n} \rightarrow \mathcal{M}f_\nu$  in  $L^2(\mathcal{U}_\nu, \mathbb{C}^4)$ .

For  $\nu \in I''$ , note that there exists a diffeomorphism  $\mathcal{F}_\nu$  from  $\mathcal{U}_\nu$  to a neighborhood  $U_\nu$  of the origin in  $\mathbb{R}^4$  such that the boundary  $\mathcal{U}_\nu \cap \partial\Omega$  is mapped to the hyperplane  $\{y_4 = 0\}$  and  $\mathcal{U}_\nu \cap \Omega$  is mapped to  $U_{\nu-}$ . Here we denote by  $y = (y_1, \dots, y_4)$  the coordinates of  $U_\nu \subset \mathbb{R}^4$  and  $y = \mathcal{F}_\nu(x)$ . Let  $L := \mathcal{F}_{\nu*}\mathcal{L}$ ,  $M := \mathcal{F}_{\nu*}\mathcal{M}$  be differential operators by pushing forward. Then by definition, we have

$$L = \sum_{k=1}^4 \sum_{j=1}^4 \mathcal{L}_j J_{jk}(y) \frac{\partial}{\partial y_k} + \mathcal{L}_0(\mathcal{F}_\nu^{-1}(y)), \quad M = \sum_{k=1}^4 \sum_{j=1}^4 \mathcal{M}_j J_{jk}(y) \frac{\partial}{\partial y_k}$$

where  $(J_{jk}(y)) = (\frac{\partial y_k}{\partial x_j})$  is the Jacobian matrix. We claim that

$$(4.24) \quad \dim \ker L_4(y) = 1 \quad \text{and} \quad \dim(\ker L_4(y) \cap \ker M_4(y)) = 0,$$

for  $y \in U_\nu$ , where

$$(4.25) \quad L_4(y) = \sum_{j=1}^4 \mathcal{L}_j J_{j4}(y), \quad M_4(y) = \sum_{j=1}^4 \mathcal{M}_j J_{j4}(y).$$

Now define functions  $h_\nu := (\mathcal{F}_\nu^{-1})^* f_\nu \in L^2(U_{\nu-}, \mathbb{C}^4)$ . Note that by the property of pulling back of distributions, we have

$$Lg = \mathcal{F}_{\nu*}\mathcal{L}(g) = (\mathcal{F}_\nu^{-1})^*(\mathcal{L}(\mathcal{F}_\nu^*g))$$

for a distribution  $g$  on  $U_\nu$ . Then by pulling (4.23) back by  $\mathcal{F}_\nu^{-1}$ , we get  $\widetilde{Lh_\nu} = \widetilde{Lh_\nu}$  on  $\mathbb{R}^4$ , and obviously  $h_\nu$  is also in the maximal domain of  $M$ . Without loss of generality, we can assume that  $\text{supp } h_\nu$  is sufficiently close to the origin as required by Proposition 4.2. So we can apply Proposition 4.2 to  $h_\nu$  to find a sequence  $h_{\nu;n} \in C_0^\infty(U_\nu, \mathbb{C}^4)$  such that their restrictions to  $U_{\nu-}$ , denoted by  $\dot{h}_{\nu;n} := h_{\nu;n}|_{U_{\nu-}}$ , are in the minimal domain of  $L$ , i.e.

$$(4.26) \quad \widetilde{Lh_{\nu;n}} = \widetilde{L\dot{h}_{\nu;n}}, \quad \text{and} \quad \dot{h}_{\nu;n} \rightarrow h_\nu, \quad L\dot{h}_{\nu;n} \rightarrow Lh_\nu, \quad M\dot{h}_{\nu;n} \rightarrow Mh_\nu \quad \text{in} \quad L^2(U_{\nu-}, \mathbb{C}^4).$$

Now pulling back to  $\mathcal{U}_\nu$  by  $\mathcal{F}_\nu$ , we get functions  $f_{\nu;n} := \mathcal{F}_\nu^*(h_{\nu;n}) \in C_0^\infty(\mathcal{U}_\nu, \mathbb{C}^4)$  satisfying

$$\widetilde{\mathcal{L}\dot{f}_{\nu;n}} = \widetilde{\mathcal{L}\dot{f}_{\nu;n}}, \quad \text{and} \quad \dot{f}_{\nu;n} \rightarrow f_\nu, \quad \mathcal{L}\dot{f}_{\nu;n} \rightarrow \mathcal{L}f_\nu, \quad \mathcal{M}\dot{f}_{\nu;n} \rightarrow \mathcal{M}f_\nu \quad \text{in} \quad L^2(\mathcal{U}_\nu, \mathbb{C}^4),$$

by pulling back (4.26), where  $\dot{f}_{\nu;n} := f_{\nu;n}|_\Omega = \mathcal{F}_\nu^*(\dot{h}_{\nu;n})$  is the restriction of  $f_{\nu;n}$  to  $\Omega$ . So by Proposition 4.1,  $\dot{f}_{\nu;n}$  is in the minimal domain of  $\mathcal{L}$ . Then the finite sum  $f_n = \sum_{\nu \in I'} f_{\nu;n} + \sum_{\nu \in I''} f_{\nu;n} \in C^\infty(\overline{\Omega}, \mathbb{C}^4)$  is also in the minimal domain of  $\mathcal{L}$ , and  $f_n|_\Omega \rightarrow f$  in  $L^2(\Omega, \mathbb{C}^4)$ .

It remains to prove the claim (4.24). For a fixed point  $y$ , write  $\xi := (J_{14}(y), \dots, J_{44}(y)) \neq 0$ . Comparing (4.21) with (4.25), we see that  $L_4$  and  $M_4$  are exactly the matrices (4.19) and (4.20) with  $\frac{\partial}{\partial x_j}$  replaced by  $\xi_j$ , respectively, i.e.

$$(4.27) \quad L_4(y) = \begin{pmatrix} \xi_3 - \mathbf{i}\xi_4 & -\xi_1 - \mathbf{i}\xi_2 & 0 & 0 \\ \frac{1}{2}(\xi_1 - \mathbf{i}\xi_2) & \frac{1}{2}(\xi_3 + \mathbf{i}\xi_4) & \frac{1}{2}(\xi_3 - \mathbf{i}\xi_4) & \frac{1}{2}(-\xi_1 - \mathbf{i}\xi_2) \\ 0 & 0 & \xi_1 - \mathbf{i}\xi_2 & \xi_3 + \mathbf{i}\xi_4 \end{pmatrix},$$

$$M_4(y) = \frac{1}{2} \begin{pmatrix} -\xi_1 + \mathbf{i}\xi_2 & -\xi_3 - \mathbf{i}\xi_4 & \xi_3 - \mathbf{i}\xi_4 & -\xi_1 - \mathbf{i}\xi_2 \end{pmatrix},$$

by (4.15) and (3.7), respectively.  $\ker L_4(y)$  is of dimension 3 for any  $0 \neq \xi \in \mathbb{R}^4$  by

$$(4.28) \quad \det \begin{pmatrix} \xi_3 - \mathbf{i}\xi_4 & -\xi_1 - \mathbf{i}\xi_2 \\ \xi_1 - \mathbf{i}\xi_2 & \xi_3 + \mathbf{i}\xi_4 \end{pmatrix} = |\xi|^2 \quad \text{and} \quad \begin{pmatrix} \xi_1 - \mathbf{i}\xi_2 & \xi_3 + \mathbf{i}\xi_4 \end{pmatrix} \text{ is nondegenerate.}$$

For  $(a_1, \dots, a_4)^t \in \ker L_4(y) \cap \ker M_4(y)$ , it is easy to see that  $\begin{pmatrix} \xi_3 - \mathbf{i}\xi_4 & -\xi_1 - \mathbf{i}\xi_2 \\ \xi_1 - \mathbf{i}\xi_2 & \xi_3 + \mathbf{i}\xi_4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mathbf{0}$  by comparing the second row of  $L_4(y)$  with  $M_4(y)$ . Therefore  $(a_1, a_2) = (0, 0)$ , and consequently  $(a_1, \dots, a_4) = (0, \dots, 0)$ . The Lemma is proved.

#### 4.3. Proof of Theorem 1.2.

**Proposition 4.3.** *For  $k = 2, 3, \dots$ , the associated Laplacian operator  $\square_\varphi$  in (1.12) is a densely-defined, closed, self-adjoint and non-negative operator on  $L_\varphi^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ .*

The proof is exactly the same as the proof of proposition 4.2.3 of [7] for  $\bar{\partial}$ -complex once we have the following estimate (4.29). See proposition 3.1 in [23] for a complete proof for the  $k$ -Cauchy-Fueter complexes on weighted  $L^2$  space over  $\mathbb{R}^{4n}$ .

*Proof of Theorem 1.2.* (1) The  $L^2$  estimate (1.17) in Theorem 1.1 implies that

$$C_0 \|g\|_\varphi^2 \leq \|\mathcal{D}_0^* g\|_\varphi^2 + \|\mathcal{D}_1 g\|_\varphi^2 = (\square_\varphi g, g)_\varphi \leq \|\square_\varphi g\|_\varphi \|g\|_\varphi,$$

for  $g \in \text{Dom}(\square_\varphi)$ , i.e.

$$(4.29) \quad C_0 \|g\|_\varphi \leq \|\square_\varphi g\|_\varphi.$$

Thus  $\square_\varphi$  is injective. This together with self-adjointness of  $\square_\varphi$  in Proposition 4.3 implies the density of the range (cf. section 2 of Chapter 8 in [17] for this general property of a densely-defined injective self-adjoint operator). For fixed  $f \in L_\varphi^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ , the complex anti-linear functional

$$\lambda_f : \square_\varphi g \longrightarrow \langle f, g \rangle_\varphi$$

is then well-defined on the dense subspace  $\mathcal{R}(\square_\varphi)$  of  $L_\varphi^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ , and is finite since

$$|\lambda_f(\square_\varphi g)| = |\langle f, g \rangle_\varphi| \leq \|f\|_\varphi \|g\|_\varphi \leq \frac{1}{C_0} \|f\|_\varphi \|\square_\varphi g\|_\varphi$$

for any  $g \in \text{Dom}(\square_\varphi)$ , by (4.29). So  $\lambda_f$  can be uniquely extended a continuous complex anti-linear functional on  $L_\varphi^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ . By the Riesz representation theorem, there exists a unique element  $h \in L_\varphi^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$  such that  $\lambda_f(k) = \langle h, k \rangle_\varphi$  for any  $k \in L_\varphi^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ , and  $\|h\|_\varphi = |\lambda_f| \leq \frac{1}{C_0} \|f\|_\varphi$ . In particular, we have

$$\langle h, \square_\varphi g \rangle_\varphi = \langle f, g \rangle_\varphi$$

for any  $g \in \text{Dom}(\square_\varphi)$ . This implies that  $h \in \text{Dom}(\square_\varphi^*)$  and  $\square_\varphi^* h = f$ . By self-adjointness of  $\square_\varphi$  in Proposition 4.3, we find that  $h \in \text{Dom}(\square_\varphi)$  and  $\square_\varphi h = f$ . We write  $h = N_\varphi f$ . Then  $\|N_\varphi f\|_\varphi \leq \frac{1}{C_0} \|f\|_\varphi$ .

(2) Since  $N_\varphi f \in \text{Dom}(\square_\varphi)$ , we have  $\mathcal{D}_0^* N_\varphi f \in \text{Dom}(\mathcal{D}_0)$ ,  $\mathcal{D}_1 N_\varphi f \in \text{Dom}(\mathcal{D}_1^*)$ , and

$$(4.30) \quad \mathcal{D}_0 \mathcal{D}_0^* N_\varphi f = f - \mathcal{D}_1^* \mathcal{D}_1 N_\varphi f$$

by  $\square_\varphi N_\varphi f = f$ . Because  $f$  and  $\mathcal{D}_0 u$  for any  $u \in \text{Dom}(\mathcal{D}_0)$  are both  $\mathcal{D}_1$ -closed ( $\mathcal{D}_1 \circ \mathcal{D}_0 = 0$  by (2.19)), the above identity implies  $\mathcal{D}_1^* \mathcal{D}_1 N_\varphi f \in \text{Dom}(\mathcal{D}_1)$  and so  $\mathcal{D}_1 \mathcal{D}_1^* \mathcal{D}_1 N_\varphi f = 0$  by  $\mathcal{D}_1$  acting on both sides of (4.30). Then

$$0 = \langle \mathcal{D}_1 \mathcal{D}_1^* \mathcal{D}_1 N_\varphi f, \mathcal{D}_1 N_\varphi f \rangle_\varphi = \|\mathcal{D}_1^* \mathcal{D}_1 N_\varphi f\|_\varphi^2,$$

i.e.  $\mathcal{D}_1^* \mathcal{D}_1 N_\varphi f = 0$ . Hence  $\mathcal{D}_0 \mathcal{D}_0^* N_\varphi f = f$  by (4.30). Moreover, we have  $\mathcal{D}_0^* N_\varphi f \perp A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$  since  $\langle v, \mathcal{D}_0^* N_\varphi f \rangle_\varphi = \langle \mathcal{D}_0 v, N_\varphi f \rangle_\varphi = 0$  for any  $v \in A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$ . The estimate (1.18) follows from

$$\|\mathcal{D}_0^* N_\varphi f\|_\varphi^2 + \|\mathcal{D}_1 N_\varphi f\|_\varphi^2 = \langle \square_\varphi N_\varphi f, N_\varphi f \rangle_\varphi \leq \frac{1}{C_0} \|f\|_\varphi^2. \quad \square$$

## 5. THE $L^2$ -ESTIMATE FOR GENERAL $k$

**5.1. The Neumann boundary condition,  $k$ -plurisubharmonicity and  $k$ -pseudoconvexity.** Let us generalize results in section 3 to general  $k$ .

**Proposition 5.1.** (1) For  $f \in C_0^1(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ ,  $F \in C_0^1(\Omega, \odot^{k-2}\mathbb{C}^2 \otimes \wedge^2\mathbb{C}^2)$ , we have

$$(5.1) \quad (\mathcal{D}_0^* f)_{A'_1 \dots A'_k} = \sum_{A=0,1} \delta_{(A'_1 f_{A'_2 \dots A'_k} A)}^A, \quad (\mathcal{D}_1^* F)_{A'_2 \dots A'_k A} = \sum_{A=0,1} \delta_{(A'_2 F_{A'_3 \dots A'_k} A)}^B.$$

(2)  $f \in C^1(\overline{\Omega}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$  if and only if

$$(5.2) \quad \sum_{A=0,1} Z_{(A'_1}^A r \cdot f_{A'_2 \dots A'_k} A) = 0 \quad \text{on } \partial\Omega$$

for any  $A_1, \dots, A'_k = 0', 1'$ ; and  $F \in C^1(\overline{\Omega}, \odot^{k-2}\mathbb{C}^2 \otimes \wedge^2\mathbb{C}^2) \cap \text{Dom}\mathcal{D}_1^*$  if and only if

$$(5.3) \quad \sum_{B=0,1} Z_{(A'_2}^B r \cdot F_{A'_3 \dots A'_k} B) = 0, \quad \text{on } \partial\Omega$$

for any  $A_2, \dots, A'_k = 0', 1'$ ,  $A = 0, 1$

*Proof.* (1) For any  $ug \in C_0^\infty(\Omega, \odot^k\mathbb{C}^2)$ ,

$$\begin{aligned} \langle \mathcal{D}_0 u, f \rangle_\varphi &= \sum_{A, A'_1 \dots A'_k} \left( Z_{A'}^{A'_1} u_{A'_1 \dots A'_k}, f_{A'_2 \dots A'_k A} \right)_\varphi = \sum_{A, A'_1 \dots A'_k} \left( u_{A'_1 \dots A'_k}, \delta_{A'_1}^A f_{A'_2 \dots A'_k A} \right)_\varphi \\ &= \sum_{A'_1 \dots A'_k} \left( u_{A'_1 \dots A'_k}, \sum_A \delta_{(A'_1 f_{A'_2 \dots A'_k} A)}^A \right)_\varphi = \langle u, \mathcal{D}_0^* f \rangle_\varphi \end{aligned}$$

by  $\delta_{A'}^A$  as the formal adjoint operator of  $Z_{A'}^{A'_1}$  in (2.6). Here we need to symmetrise primed indices in  $\delta_{A'_1}^A f_{A'_2 \dots A'_k A}$  by using Proposition 2.2, since only after symmetrisation it becomes a  $\odot^k\mathbb{C}^2$ -valued function.

(2) If  $f \in C^1(\overline{\Omega}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$ , then for any  $u \in C^2(\overline{\Omega}, \odot^k\mathbb{C}^2)$  we have

$$\begin{aligned} \langle \mathcal{D}_0 u, f \rangle_\varphi &= \sum_{A, A'_1 \dots A'_k} \left( Z_{A'}^{A'_1} u_{A'_1 \dots A'_k}, f_{A'_2 \dots A'_k A} \right)_\varphi = \sum_{A, A'_1 \dots A'_k} \left( u_{A'_1 \dots A'_k}, \delta_{A'_1}^A f_{A'_2 \dots A'_k A} \right)_\varphi + \mathcal{B}_0 \\ &= \sum_{A'_1 \dots A'_k} \left( u_{A'_1 \dots A'_k}, \sum_A \delta_{(A'_1 f_{A'_2 \dots A'_k} A)}^A \right)_\varphi + \mathcal{B}_0 \end{aligned}$$

by applying Stokes' formula (2.8) and using symmetrisation by Proposition 2.2, with the boundary term

$$\mathcal{B}_0 := - \sum_{A, A'_1, \dots} \int_{\partial\Omega} u_{A'_1 \dots A'_k} \overline{Z_{A'_1}^A r \cdot f_{A'_2 \dots A'_k A}} e^{-\varphi} dS = - \int_{\partial\Omega} \sum_{A'_1, \dots} u_{A'_1 \dots A'_k} \cdot \overline{\sum_A Z_{(A'_1}^A r \cdot f_{A'_2 \dots A'_k} A)} e^{-\varphi} dS,$$

by using symmetrisation by Proposition 2.2 again. Thus as in the case  $k = 2$ ,  $\langle \mathcal{D}_0 u, f \rangle_\varphi = \langle u, \mathcal{D}_0^* f \rangle_\varphi$  if and only if the boundary term (5.1) vanishes for any  $\odot^k\mathbb{C}^2$ -valued function  $u$ , i.e. (5.2) holds.

Now for  $h \in C^1(\bar{\Omega}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ , we have

$$\begin{aligned} \langle \mathcal{D}_1 h, F \rangle_\varphi &= \sum_{B, A, A'_2, \dots, A'_k} \left( Z_{[B}^{A'_2} h_{A]A'_2 \dots A'_k}, F_{A'_3 \dots A'_k BA} \right)_\varphi = \sum_{B, A, A'_2, \dots, A'_k} \left( Z_B^{A'_2} h_{AA'_2 \dots A'_k}, f_{A'_3 \dots A'_k BA} \right)_\varphi \\ &= \sum_{A, B, A'_2, \dots, A'_k} \left( h_{A'_2 \dots A'_k A}, \delta_{A'_2}^B F_{A'_3 \dots A'_k BA} \right)_\varphi + \mathcal{B}'_0 \\ &= \sum_{A, A'_2, \dots, A'_k} \left( h_{A'_2 \dots A'_k A}, \sum_B \delta_{(A'_2}^B F_{A'_3 \dots A'_k) BA} \right)_\varphi + \mathcal{B}'_0 \end{aligned}$$

by dropping antisymmetrisation by (2.12), applying Stokes' formula (2.8) as above and using symmetrisation. Thus  $\langle \mathcal{D}_1 h, F \rangle_\varphi = \langle h, \mathcal{D}_1^* F \rangle_\varphi$  if and only if the boundary term

$$\begin{aligned} \mathcal{B}'_0 &= - \int_{\partial\Omega} \sum_{B, A, A'_2, \dots} h_{A'_2 \dots A'_k A} \overline{Z_{A'_2}^B r \cdot F_{A'_3 \dots A'_k BA}} e^{-\varphi} dS \\ &= - \int_{\partial\Omega} \sum_{A, A'_2, \dots} h_{A'_2 \dots A'_k A} \sum_B \overline{Z_{(A'_2}^B r \cdot F_{A'_3 \dots A'_k) BA}} e^{-\varphi} dS = 0 \end{aligned}$$

for any  $h$ , i.e. (5.3) holds.  $\mathcal{D}_1^*$  in (5.1) also follows.  $\square$

For a  $C^2$  real function  $\varphi$ , define

$$(5.4) \quad \mathcal{L}_k(\varphi; \xi)(x) := -k \sum_{A, B, A'_1, \dots, A'_k} Z_B^{A'_1} Z_{(A'_1}^A \varphi(x) \cdot \xi_{A'_2 \dots A'_k A} \overline{\xi_{A'_2 \dots A'_k B}}$$

for any  $(\xi_{A'_2 \dots A'_k A}) \in \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2$ . A  $C^2$  real function  $\varphi$  on  $\Omega$  is called (*strictly*)  $k$ -*plurisubharmonic* if there exists a constant some  $c \geq 0$  ( $c > 0$ ) such that

$$\mathcal{L}_k(\varphi; \xi)(x) \geq c|\xi|^2,$$

for any  $x \in \Omega$ ,  $\xi \in \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2$ . A domain in  $\mathbb{R}^4$  is called (*strictly*)  $k$ -*pseudoconvex* if there exists a defining function  $r$  and a constant  $c \geq 0$  ( $c > 0$ ) such that

$$(5.5) \quad \mathcal{L}_k(r; \xi)(x) \geq c|\xi|^2, \quad x \in \partial\Omega,$$

for any  $(\xi_{A'_2 \dots A'_k A}) \in \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2$  satisfying

$$(5.6) \quad \sum_{A=0,1} Z_{(A'_1}^A r(x) \cdot \xi_{A'_2 \dots A'_k A} = 0, \quad x \in \partial\Omega,$$

for any  $A'_1 \dots A'_k = 0', 1'$ . The space of vectors  $\xi$  satisfying (5.6) is of dimension  $\geq 2k - (k+1) = k-1$ , since there are  $k+1$  equations in (5.6).

Recall that if  $(u_{A'_1 \dots A'_p}) \in \otimes^p \mathbb{C}^2$  is symmetric in  $A'_2 \dots A'_p$ , then we have

$$(5.7) \quad u_{(A'_1 \dots A'_p)} = \frac{1}{k} \left( u_{A'_1 A'_2 \dots A'_p} + \dots + u_{A'_s A'_1 \dots \widehat{A'_s} \dots A'_p} + \dots + u_{A'_p A'_1 \dots A'_{p-1}} \right),$$

by definition (2.14) of symmetrisation.

**Proposition 5.2.** *Suppose that  $\chi$  is an increasing smooth convex function over  $[0, \infty)$ . Then for  $\psi(x) = \chi(\varphi(x))$ , we have*

$$(5.8) \quad \mathcal{L}_k(\psi; \xi) \geq \chi'(\varphi) \mathcal{L}_k(\varphi; \xi)$$

for any  $\xi \in \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2$ . In particular  $\psi$  is  $k$ -plurisubharmonic if  $\varphi$  is.

*Proof.* For any  $(\xi_{A'_2 \dots A'_k A}) \in \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2$ , it is easy to see that

$$\begin{aligned}
\mathcal{L}_k(\psi; \xi) &= - \sum_{A, B, A'_1, \dots, A'_k} Z_B^{A'_1} Z_{(A'_1}^A \psi \cdot \xi_{A'_2 \dots A'_k) A} \overline{\xi_{A'_2 \dots A'_k B}} \\
&= - \chi'(\varphi(x)) \sum_{A, B, A'_1, \dots, A'_k} Z_B^{A'_1} Z_{(A'_1}^A \varphi \cdot \xi_{A'_2 \dots A'_k) A} \overline{\xi_{A'_2 \dots A'_k B}} \\
&\quad - \chi''(\varphi(x)) \sum_{A, B, A'_1, \dots, A'_k} Z_B^{A'_1} \varphi \cdot Z_{(A'_1}^A \varphi \cdot \xi_{A'_2 \dots A'_k) A} \overline{\xi_{A'_2 \dots A'_k B}} \\
&= \chi'(\varphi(x)) \mathcal{L}_k(\varphi; \xi) + \chi''(\varphi(x)) \sum_{A'_1, \dots, A'_k} \sum_B \overline{Z_{A'_1}^B \varphi \cdot \xi_{A'_2 \dots A'_k B}} \cdot \sum_A Z_{(A'_1}^A \varphi \cdot \xi_{A'_2 \dots A'_k) A} \\
&= \chi'(\varphi(x)) \mathcal{L}_k(\varphi; \xi) + \chi''(\varphi(x)) \sum_{A'_1, \dots, A'_k} \left| \sum_B Z_{(A'_1}^B \varphi \cdot \xi_{A'_2 \dots A'_k) B} \right|^2
\end{aligned}$$

by  $-Z_B^{A'} \varphi = \overline{Z_{A'}^B \varphi}$  for real  $\varphi$  by (2.3) and using symmetrisation by Proposition 2.2. The result follows.  $\square$

**Proposition 5.3.**  *$k$ -pseudoconvexity of a domain is independent of the choice of the defining function.*

*Proof.* Suppose that  $r$  and  $\tilde{r}$  are both defining functions of the domain  $\Omega$ . Then  $\tilde{r}(x) = \mu(x)r(x)$  for some nonvanishing function  $\mu > 0$  near  $\partial\Omega$ . Note that  $Z_{A'}^A \tilde{r} = \mu \cdot Z_{A'}^A r$  on the boundary. It is obvious that for any  $x \in \partial\Omega$ ,  $A'_1 \dots A'_k = 0', 1', \xi \in \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2$  satisfies  $\sum_{A=0,1} Z_{(A'_1}^A \tilde{r}(x) \cdot \xi_{A'_2 \dots A'_k) A} = 0$ , if and only if it satisfies  $\sum_{A=0,1} Z_{(A'_1}^A r(x) \cdot \xi_{A'_2 \dots A'_k) A} = 0$ . So the boundary condition (5.6) for  $\xi$  is independent of the choice of the defining function. Then for  $x \in \partial\Omega$  and  $\xi$  satisfying the boundary condition (5.6), we have

$$\begin{aligned}
\mathcal{L}_k(\tilde{r}; \xi)(x) &= - \sum_{A, B, A'_1, \dots, A'_k} \sum_{s=1}^k Z_B^{A'_1} Z_{A'_s}^A (\mu(x)r(x)) \cdot \xi_{A'_1 \dots \widehat{A'_s} \dots A'_k A} \overline{\xi_{A'_2 \dots A'_k B}} \\
&= \mu(x) \mathcal{L}(r; \xi)(x) + r(x) \mathcal{L}(\mu; \xi) + \Sigma' + \Sigma''
\end{aligned}$$

(the second term vanishes on the boundary) by (5.7) with

$$\begin{aligned}
\Sigma' &= - \sum_{A, B, A'_1, \dots, A'_k} \sum_{s=1}^k Z_B^{A'_1} \mu(x) \cdot Z_{A'_s}^A r(x) \cdot \xi_{A'_1 \dots \widehat{A'_s} \dots A'_k A} \overline{\xi_{A'_2 \dots A'_k B}} \\
&= -k \sum_{B, A'_1, \dots, A'_k} Z_B^{A'_1} \mu(x) \cdot \overline{\xi_{A'_2 \dots A'_k B}} \cdot \sum_A Z_{(A'_1}^A r(x) \xi_{A'_2 \dots A'_k) A} = 0,
\end{aligned}$$

by the condition (5.6) for  $\xi$  on the boundary, and by  $Z_B^{A'} = -\overline{Z_{A'}^B}$

$$\begin{aligned}
\Sigma'' &= - \sum_{A, B, A'_1, \dots, A'_k} Z_B^{A'_1} r(x) \overline{\xi_{A'_2 \dots A'_k B}} \cdot \sum_{s=1}^k Z_{A'_s}^A \mu(x) \cdot \xi_{A'_1 \dots \widehat{A'_s} \dots A'_k A} \\
&= k \sum_{A'_1, \dots, A'_k} \sum_B \overline{Z_{A'_1}^B r(x) \xi_{A'_2 \dots A'_k B}} \cdot \sum_A Z_{(A'_1}^A \mu(x) \cdot \xi_{A'_2 \dots A'_k) A} \\
&= k \sum_{A'_1, \dots, A'_k} \sum_B \overline{Z_{(A'_1}^B r(x) \xi_{A'_2 \dots A'_k) B}} \cdot \sum_A Z_{(A'_1}^A \mu(x) \cdot \xi_{A'_2 \dots A'_k) A} = 0,
\end{aligned}$$

for  $\xi$  satisfying the boundary condition (5.6), by using symmetrisation by Proposition 2.2. At last we get  $\mathcal{L}_k(\tilde{r}; \xi) = \mu \mathcal{L}_k(r; \xi)$  on the boundary for  $\xi$  satisfying the condition (5.6). The result follows.  $\square$

$k$ -plurisubharmonic functions and  $k$ -pseudoconvex domains are abundant by the following examples, and by definition a small perturbation of a strictly  $k$ -plurisubharmonic function is still strictly  $k$ -plurisubharmonic.

**Proposition 5.4.**  $r_1(x) = x_1^2 + x_2^2$  and  $r_2(x) = x_3^2 + x_4^2$  are both strictly  $k$ -plurisubharmonic, and  $\chi_1(r_1)$ ,  $\chi_2(r_2)$  and their sum are all (strictly)  $k$ -plurisubharmonic for any increasing smooth (strictly) convex functions  $\chi_1$  and  $\chi_2$  over  $[0, \infty)$ .

*Proof.* As in the case  $k = 2$ , note that

$$\mathcal{L}_k(r; \xi)(x) = \sum_{A, B, A'_1, \dots, A'_k} Z_B^{A'_1} \overline{Z_A^{A'_1}} r \cdot \xi_{A'_2 \dots A'_k A} \overline{\xi_{A'_2 \dots A'_k B}} + \sum_{s=2}^k \sum_{A, B, A'_1, \dots, A'_k} Z_B^{A'_1} \overline{Z_A^{A'_s}} r \cdot \xi_{A'_1 \dots \widehat{A'_s} \dots A'_k A} \overline{\xi_{A'_2 \dots A'_k B}},$$

and so according to  $(A, B) = (0, 0), (1, 1), (1, 0)$  and  $(0, 1)$ , we get

$$\begin{aligned} \mathcal{L}_k(r_1; \xi) &= \sum_{A'_2, \dots, A'_k} Z_0^{1'} \overline{Z_0^{1'}} r_1 \cdot |\xi_{A'_2 \dots A'_k 0}|^2 + \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_0^{1'} \overline{Z_0^{1'}} r_1 \cdot \xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 0} \overline{\xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 0}} \\ &+ \sum_{A'_2, \dots, A'_k} Z_1^{0'} \overline{Z_1^{0'}} r_1 \cdot |\xi_{A'_2 \dots A'_k 1}|^2 + \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_1^{0'} \overline{Z_1^{0'}} r_1 \cdot \xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 1} \overline{\xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 1}} \\ &+ \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_0^{1'} \overline{Z_1^{0'}} r_1 \cdot \xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 1} \overline{\xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 0}} \\ &+ \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_1^{0'} \overline{Z_0^{1'}} r_1 \cdot \xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 0} \overline{\xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 1}} \\ &= 4 \sum_{A, A'_2, \dots, A'_k} |\xi_{A'_2 \dots A'_k A}|^2 + 4(k-1) \sum_{B'_3, \dots, B'_k} \left( |\xi_{1' B'_3 \dots B'_k 0}|^2 + |\xi_{0' B'_3 \dots B'_k 1}|^2 \right) \geq 4|\xi|^2 \end{aligned}$$

by (3.8). Similarly  $Z_0^{1'}$  and  $Z_1^{0'}$  are independent of  $x_3$  and  $x_4$  by (3.7), and so

$$\begin{aligned} \mathcal{L}_k(r_2; \xi) &= \sum_{A'_2, \dots, A'_k} Z_0^{0'} \overline{Z_0^{0'}} r_2 \cdot |\xi_{A'_2 \dots A'_k 0}|^2 + \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_0^{0'} \overline{Z_0^{0'}} r_2 \cdot \xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 0} \overline{\xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 0}} \\ &+ \sum_{A'_2, \dots, A'_k} Z_1^{1'} \overline{Z_1^{1'}} r_2 \cdot |\xi_{A'_2 \dots A'_k 1}|^2 + \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_1^{1'} \overline{Z_1^{1'}} r_2 \cdot \xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 1} \overline{\xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 1}} \\ &+ \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_0^{0'} \overline{Z_1^{1'}} r_2 \cdot \xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 1} \overline{\xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 0}} \\ &+ \sum_{s=2}^k \sum_{A'_2, \dots, \widehat{A'_s}, \dots} Z_1^{1'} \overline{Z_0^{0'}} r_2 \cdot \xi_{1' A'_2 \dots \widehat{A'_s} \dots A'_k 0} \overline{\xi_{0' A'_2 \dots \widehat{A'_s} \dots A'_k 1}} \\ &= 4 \sum_{A, A'_2, \dots, A'_k} |\xi_{A'_2 \dots A'_k A}|^2 + 4(k-1) \sum_{B'_3, \dots, B'_k} \left( |\xi_{0' B'_3 \dots B'_k 0}|^2 + |\xi_{1' B'_3 \dots B'_k 1}|^2 \right) \geq 4|\xi|^2 \end{aligned}$$

by

$$Z_0^{0'} \overline{Z_0^{0'}} r_2 = 4 = Z_1^{1'} \overline{Z_1^{1'}} r_2, \quad \text{and} \quad Z_1^{1'} \overline{Z_0^{0'}} r_2 = 0 = Z_0^{0'} \overline{Z_1^{1'}} r_2,$$

since

$$Z_0^{0'} \overline{Z_0^{0'}} = \partial_{x_3}^2 + \partial_{x_4}^2 = Z_1^{1'} \overline{Z_1^{1'}}, \quad \text{and} \quad Z_1^{1'} \overline{Z_0^{0'}} = \partial_{x_3}^2 - \partial_{x_4}^2 - 2\mathbf{i}\partial_{x_3}\partial_{x_4},$$

which follows from (3.7). The result follows.  $\square$

**Lemma 5.1.**  $C^\infty(\overline{\Omega}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2) \cap \text{Dom}(\mathcal{D}_0^*)$  is dense in  $\text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$ .

*Proof.* If we use notations in (2.21) to identify linear spaces (2.20), the proof is exactly same as the case  $k = 2$  except for the operators replaced by

$$\mathcal{L}f = [\widehat{\mathcal{L}} - \widehat{\mathcal{L}}\varphi]f, \quad \widehat{\mathcal{L}} = \begin{pmatrix} Z_0^{0'} & Z_1^{1'} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{k}Z_1^{0'} & \frac{1}{k}Z_1^{1'} & \frac{k-1}{k}Z_0^{0'} & \frac{k-1}{k}Z_1^{1'} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{2}{k}Z_1^{0'} & \frac{2}{k}Z_1^{1'} & \frac{k-2}{k}Z_0^{0'} & \frac{k-2}{k}Z_1^{1'} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{3}{k}Z_1^{0'} & \frac{3}{k}Z_1^{1'} & \frac{k-3}{k}Z_0^{0'} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

(cf. (5.17) in the sequel), where  $\widehat{\mathcal{L}}$  is a  $(k+1) \times (2k)$ -matrix valued differential operator with constant complex coefficients, and

$$\widehat{\mathcal{M}} = \frac{1}{2} \begin{pmatrix} -Z_1^{0'} & Z_0^{0'} & -Z_1^{1'} & Z_0^{1'} & 0 & 0 & 0 & \cdots \\ 0 & 0 & -Z_1^{0'} & Z_0^{0'} & -Z_1^{1'} & Z_0^{1'} & 0 & \cdots \\ 0 & 0 & 0 & 0 & -Z_1^{0'} & Z_0^{0'} & -Z_1^{1'} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

as a  $(k+1) \times (2k)$ -matrix valued differential operator with constant complex coefficients (note that here  $\mathcal{L}$  as the formal adjoint operator of  $\mathcal{D}_0$  is different from that in [6] [21], because we use the inner product (2.15) different from that in [6] [21] even when  $\varphi = 0$ ). With  $\frac{\partial}{\partial x_j}$  replaced by  $\xi_j$ , we have

$$L_4(y) = \begin{pmatrix} \xi_3 - \mathbf{i}\xi_4 & -\xi_1 - \mathbf{i}\xi_2 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{k}(\xi_1 - \mathbf{i}\xi_2) & \frac{1}{k}(\xi_3 + \mathbf{i}\xi_4) & \frac{k-1}{k}(\xi_3 - \mathbf{i}\xi_4) & \frac{k-1}{k}(-\xi_1 - \mathbf{i}\xi_2) & 0 & 0 & \cdots \\ 0 & 0 & \frac{2}{k}(\xi_1 - \mathbf{i}\xi_2) & \frac{2}{k}(\xi_3 + \mathbf{i}\xi_4) & \frac{k-2}{k}(\xi_3 - \mathbf{i}\xi_4) & \frac{k-2}{k}(-\xi_1 - \mathbf{i}\xi_2) & \cdots \\ 0 & 0 & 0 & 0 & \frac{3}{k}(\xi_1 - \mathbf{i}\xi_2) & \frac{3}{k}(\xi_3 + \mathbf{i}\xi_4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

by (4.15), which is obviously of rank  $k+1$  for any  $0 \neq \xi \in \mathbb{R}^4$  by the same reason as in the case  $k = 2$  by (4.28), and so its kernel is of dimension  $k-1$  for any  $y$ . With  $\frac{\partial}{\partial x_j}$  replaced by  $\xi_j$ , we have

$$M_4(y) = \frac{1}{2} \begin{pmatrix} -\xi_1 + \mathbf{i}\xi_2 & -\xi_3 - \mathbf{i}\xi_4 & \xi_3 - \mathbf{i}\xi_4 & -\xi_1 - \mathbf{i}\xi_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\xi_1 + \mathbf{i}\xi_2 & -\xi_3 - \mathbf{i}\xi_4 & \xi_3 - \mathbf{i}\xi_4 & -\xi_1 - \mathbf{i}\xi_2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\xi_1 + \mathbf{i}\xi_2 & -\xi_3 - \mathbf{i}\xi_4 & \xi_3 - \mathbf{i}\xi_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

by (3.7).  $\dim(\ker L_4(y) \cap \ker M_4(y)) = 0$  by the same reason as in the case  $k = 2$ .  $\square$

**5.2. Proof of Theorem 1.1 for  $k > 2$ .** By the density Lemma 5.1, it is sufficient to show the  $L^2$  estimate (1.17) for  $f \in C^\infty(\overline{\Omega}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$  satisfying the boundary condition (5.2). By expanding symmetrisation in terms of (5.7), using commutators (4.2) and Stokes' formula (2.9), we get

$$\begin{aligned}
(5.9) \quad k\langle \mathcal{D}_0^* f, \mathcal{D}_0^* f \rangle_\varphi &= k\langle \mathcal{D}_0 \mathcal{D}_0^* f, f \rangle_\varphi = k \sum_{B, A'_1, \dots, A'_k} \left( Z_B^{A'_1} \sum_A \delta_{(A'_1 f_{A'_2 \dots A'_k})A}^A, f_{BA'_2 \dots A'_k} \right)_\varphi \\
&= \sum_{A, B, A'_1, \dots, A'_k} \left( Z_B^{A'_1} \delta_{A'_1}^A f_{A'_2 \dots A'_k A} + \sum_{s=2}^k Z_B^{A'_1} \delta_{A'_s}^A f_{A'_1 \dots \widehat{A'_s} \dots A'_k A}, f_{A'_2 \dots A'_k B} \right)_\varphi \\
&= \sum_{A, B, A'_1, \dots, A'_k} \left\{ \left( \delta_{A'_1}^A Z_B^{A'_1} f_{A'_2 \dots A'_k A} + \sum_{s=2}^k \delta_{A'_s}^A Z_B^{A'_1} f_{A'_1 \dots \widehat{A'_s} \dots A'_k A}, f_{A'_2 \dots A'_k B} \right)_\varphi \right. \\
&\quad \left. + \left( [Z_B^{A'_1}, \delta_{A'_1}^A] f_{A'_2 \dots A'_k A} + \sum_{s=2}^k [Z_B^{A'_1}, \delta_{A'_s}^A] f_{A'_1 \dots \widehat{A'_s} \dots A'_k A}, f_{A'_2 \dots A'_k B} \right)_\varphi \right\} \\
&= \sum_{A, B, A'_1, \dots, A'_k} \left( Z_B^{A'_1} f_{A'_2 \dots A'_k A}, Z_A^{A'_1} f_{A'_2 \dots A'_k B} \right)_\varphi \\
&\quad + \sum_{A, B, A'_1, \dots, A'_k} \sum_{s=2}^k \left( Z_B^{A'_1} f_{A'_1 \dots \widehat{A'_s} \dots A'_k A}, Z_A^{A'_s} f_{A'_2 \dots A'_k B} \right)_\varphi + \mathcal{B}_k + \mathcal{C}_k \\
&=: \Sigma_1 + \Sigma_2 + \mathcal{B}_k + \mathcal{C}_k
\end{aligned}$$

where the commutator term is

$$\begin{aligned}
(5.10) \quad \mathcal{C}_k &:= - \sum_{A, B, A'_1, \dots, A'_k} \left( Z_B^{A'_1} Z_{A'_1}^A \varphi(x) \cdot f_{A'_2 \dots A'_k A} + \sum_{s=2}^k Z_B^{A'_1} Z_{A'_s}^A \varphi(x) \cdot f_{A'_1 \dots \widehat{A'_s} \dots A'_k A}, f_{A'_2 \dots A'_k B} \right)_\varphi \\
&= \int_\Omega \mathcal{L}_k(\varphi; f(x))(x) e^{-\varphi(x)} dV \geq c \|f\|_\varphi^2
\end{aligned}$$

by commutators (4.2) and assumption (1.15), and the boundary term is

$$(5.11) \quad \mathcal{B}_k := \sum_{A, B, A'_1, \dots, A'_k} \int_{\partial\Omega} \sum_{s=1}^k Z_{A'_s}^A r \cdot Z_B^{A'_1} f_{A'_1 \dots \widehat{A'_s} \dots A'_k A} \cdot \overline{f_{A'_2 \dots A'_k B}} e^{-\varphi} dS.$$

This boundary term can also be handled by Morrey's technique. Since  $\sum_{A=0,1} Z_{(A'_1}^B r \cdot f_{A'_2 \dots A'_k)A} = 0$  vanishing on the boundary for fixed  $A'_1 \dots A'_k = 0', 1'$ , there exists  $(k+1)$  functions  $\lambda_{A'_1 \dots A'_k} = \lambda_{(A'_1 \dots A'_k)}$  such that for  $x$  near  $\partial\Omega$ ,

$$k \sum_{A=0,1} Z_{(A'_1}^A r(x) \cdot f_{A'_2 \dots A'_k)A} = \lambda_{A'_1 \dots A'_k} \cdot r(x).$$

Now differentiate this equation by the complex vector field  $Z_B^{A'}$  to get

$$\sum_{A=0,1} \left\{ k Z_B^{A'_1} Z_{(A'_1}^A r \cdot f_{A'_2 \dots A'_k)A} + \sum_{s=1}^k Z_{A'_s}^A r \cdot Z_B^{A'_1} f_{A'_1 \dots \widehat{A'_s} \dots A'_k A} \right\} = Z_B^{A'_1} \lambda_{A'_1 \dots A'_k} \cdot r + \lambda_{A'_1 \dots A'_k} Z_B^{A'_1} r.$$



Then multiplying it by  $\overline{f_{A'_2 \dots A'_k B}}$  and taking summation over  $B, A'_1 \dots A'_k$ , we get that

$$\begin{aligned}
 (5.12) \quad & -\mathcal{L}_k(r; f(x))(x) + \sum_{A, B, A'_1 \dots A'_k} \sum_{s=1}^k Z_{A'_s}^A r(x) \cdot Z_B^{A'_1} f_{A'_1 \dots \widehat{A'_s} \dots A'_k A} \overline{f_{A'_2 \dots A'_k B}} \\
 & = - \sum_{A, B, A'_1 \dots A'_k} \lambda_{A'_1 \dots A'_k} \cdot \overline{Z_{A'_1}^B r(x) \cdot f_{A'_2 \dots A'_k B}} \\
 & = \sum_{A'_1 \dots A'_k} \left( \lambda_{A'_1 \dots A'_k}(x) \cdot \sum_B \overline{Z_{(A'_1}^B r(x) \cdot f_{A'_2 \dots A'_k) B}} \right) = 0
 \end{aligned}$$

on the boundary  $\partial\Omega$ , by using  $r(x)|_{\partial\Omega} = 0$ , symmetrisation by (2.16) and the boundary condition (5.2) for  $f$ , where  $\lambda$  is symmetric in the primed indices. Apply (5.12) to the boundary term (5.11) to get

$$(5.13) \quad \mathcal{B}_k = \int_{\partial\Omega} \mathcal{L}_k(r; f(x)) e^{-\varphi} dS \geq 0$$

by the pseudoconvexity of  $r$  and  $f$  satisfying the boundary condition (5.2).

Now for the second sum of (5.9), we have

$$\begin{aligned}
 (5.14) \quad \Sigma_2 &= \sum_{s=2}^k \sum_{A, B, A'_1, \dots, A'_k} \left( Z_B^{A'_1} f_{A A'_1 \dots \widehat{A'_s} \dots A'_k}, Z_A^{A'_s} f_{B A'_s A'_2 \dots \widehat{A'_s} \dots A'_k} \right)_\varphi \\
 &= (k-1) \sum_{A, B, B'_3, \dots, B'_k} \left( \sum_{A'} Z_B^{A'} f_{A A' B'_3 \dots B'_k}, \sum_{B'} Z_A^{B'} f_{B B' B'_3 \dots B'_k} \right)_\varphi \\
 &= (k-1) \sum_{A, B, B'_3, \dots, B'_k} \left\{ \left\| \sum_{A'} Z_A^{A'} f_{B A' B'_3 \dots B'_k} \right\|_\varphi^2 - 2 \left\| \sum_{A'} Z_{[A}^{A'} f_{B] A' B'_3 \dots B'_k} \right\|_\varphi^2 \right\} \\
 &= (k-1) \sum_{A, B, B'_3, \dots, B'_k} \left\| \sum_{A'} Z_A^{A'} f_{B A' B'_3 \dots B'_k} \right\|_\varphi^2 - 2(k-1) \|\mathcal{D}_1 f\|_\varphi^2
 \end{aligned}$$

by relabeling indices and applying Lemma 2.1 (3). By applying Lemma 2.1 (3) again and using (4.8), we get

$$\begin{aligned}
 (5.15) \quad \Sigma_1 &= \sum_{A, B, A'_1, \dots, A'_k} \left\{ \left\| Z_A^{A'_1} f_{B A'_2 \dots A'_k} \right\|_\varphi^2 - 2 \left\| Z_{[A}^{A'_1} f_{B] A'_2 \dots A'_k} \right\|_\varphi^2 \right\} \\
 &= \sum_{A, B, A'_1, \dots, A'_k} \left\| Z_A^{A'_1} f_{B A'_2 \dots A'_k} \right\|_\varphi^2 - 4 \sum_{A'_1, \dots, A'_k} \left\| Z_{A'_1}^{A'_1} [0 f_1]_{A'_2 \dots A'_k} \right\|_\varphi^2.
 \end{aligned}$$

Substituting (5.10) and (5.13)-(5.15) into (5.9), we get the estimate

$$(5.16) \quad k \|\mathcal{D}_0^* f\|_\varphi^2 + 2(k-1) \|\mathcal{D}_1 f\|_\varphi^2 \geq c \|f\|_\varphi^2 - \sum_{A'_1, \dots, A'_k} \left\| 2 Z_{A'_1}^{A'_1} [0 f_1]_{A'_2 \dots A'_k} \right\|_\varphi^2.$$

Now fix  $A'_1, \dots, A'_k$ . *Case i:*  $A'_1 + \dots + A'_k = l \neq 0, k$  and  $A'_1 = 0'$ . We can assume  $A'_k = 1$  without loss of generality. It follows from (5.7) that

$$\begin{aligned}
 \sum_A Z_{(0' f_{0' \dots 0' \underbrace{1' \dots 1'}_l)A}^A &= \frac{k-l}{k} \sum_A Z_{0' f_{0' \dots 0' \underbrace{1' \dots 1'}_l)A}^A + \frac{l}{k} \sum_A Z_{1' f_{0' \dots 0' \underbrace{1' \dots 1'}_{l-1})A}^A \\
 (5.17) \quad &= -\frac{2(k-l)}{k} Z_{0' [0 f_1] 0' \dots 0' \underbrace{1' \dots 1'}_l} - \frac{2l}{k} Z_{1' [0 f_1] 0' \dots 0' \underbrace{1' \dots 1'}_{l-1}}
 \end{aligned}$$

by  $f$  symmetric in the primed indices and using Proposition 2.2. Then

$$\begin{aligned}
 2Z_{A'_1 [0 f_1] A'_2 \dots A'_k} &= 2Z_{0' [0 f_1] 0' \dots 0' \underbrace{1' \dots 1'}_l} \\
 &= \frac{2(k-l)}{k} Z_{0' [0 f_1] 0' \dots 0' \underbrace{1' \dots 1'}_l} + \frac{2l}{k} Z_{1' [0 f_1] 0' \dots 0' \underbrace{1' \dots 1'}_{l-1}} \\
 &\quad + \frac{2l}{k} Z_{0' [0 f_1] 0' \dots 0' \underbrace{1' \dots 1'}_l} - \frac{2l}{k} Z_{1' [0 f_1] 0' \dots 0' \underbrace{1' \dots 1'}_{l-1}} \\
 &= -\sum_A Z_{(0' f_{0' \dots 0' \underbrace{1' \dots 1'}_l)A}^A - \frac{2l}{k} \sum_{A'} Z_{[0 f_1] A' 0' \dots 0' \underbrace{1' \dots 1'}_{l-1}}^{A'} \\
 &= -(\mathcal{D}_0^* f)_{A'_1 \dots A'_k} - \sum_A Z_{(A'_1 \varphi \cdot f_{A'_2 \dots A'_k)A}^A - \frac{2l}{k} (\mathcal{D}_1 f)_{01 A'_2 \dots A'_{k-1}}
 \end{aligned}$$

by using Lemma 2.2 again, (5.17) and  $f$  symmetric in the primed indices.

*Case ii:*  $A'_1 + \dots + A'_k = k-l \neq 0, k$  and  $A'_1 = 1'$ . We have the similar identity (we can assume  $A'_k = 0$  without loss of generality) by

$$\begin{aligned}
 2Z_{A'_1 [0 f_1] A'_2 \dots A'_k} &= 2Z_{1' [0 f_1] 1' \dots 1' \underbrace{0' \dots 0'}_l} \\
 &= \frac{2(k-l)}{k} Z_{1' [0 f_1] 1' \dots 1' \underbrace{0' \dots 0'}_l} + \frac{2l}{k} Z_{0' [0 f_1] 1' \dots 1' \underbrace{0' \dots 0'}_{l-1}} \\
 &\quad + \frac{2l}{k} Z_{1' [0 f_1] 1' \dots 1' \underbrace{0' \dots 0'}_l} - \frac{2l}{k} Z_{0' [0 f_1] 1' \dots 1' \underbrace{0' \dots 0'}_{l-1}} \\
 &= -(\mathcal{D}_0^* f)_{A'_1 \dots A'_k} - \sum_A Z_{(A'_1 \varphi \cdot f_{A'_2 \dots A'_k)A}^A + \frac{2l}{k} (\mathcal{D}_1 f)_{01 A'_2 \dots A'_{k-1}}.
 \end{aligned}$$

*Case iii:*  $A'_1 = \dots = A'_k = 0'$  or  $1'$ . We have

$$2Z_{0' [0 f_1] 0' \dots 0'} = -\sum_A Z_{(0' f_{0' \dots 0'})A}^A = -(\mathcal{D}_0^* f)_{0' \dots 0'} - \sum_A Z_{(0' \varphi \cdot f_{0' \dots 0'})A}^A,$$

and similar identity holds for  $A'_1 = \dots = A'_k = 1'$ . We can use  $|(a_1 + \dots + a_k)/k|^2 \leq (|a_1|^2 + \dots + |a_k|^2)/k$  and the Cauchy-Schwarz inequality to control the norm of  $\sum_A Z_{(A'_1 \varphi \cdot f_{A'_2 \dots A'_k)A}^A$  by  $2\|d\varphi\|_\infty^2 \|f\|_\varphi^2$  as in (4.13)-(4.14) for the case  $k=2$ . Then by using  $|a+b|^2 \leq 2(|a|^2 + |b|^2)$  twice, we get

$$(5.18) \quad \sum_{A'_1, \dots, A'_k} \left\| 2Z_{A'_1 [0 f_1] A'_2 \dots A'_k} \right\|_\varphi^2 \leq 4\|\mathcal{D}_0^* f\|_\varphi^2 + 4\|d\varphi\|_\infty^2 \|f\|_\varphi^2 + 8\|\mathcal{D}_1 f\|_\varphi^2.$$

Now substitute (5.18) into (5.16) to get the estimate

$$(k+4)\|\mathcal{D}_0^*f\|_\varphi^2 + (2k+6)\|\mathcal{D}_1f\|_\varphi^2 \geq (c-4\|d\varphi\|_\infty^2)\|f\|_\varphi^2.$$

The estimate (1.17) is proved.  $\square$

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